**Complex Analysis** 

Answer all questions completely. Calculators may not be used. Notation:  $\mathbb{D} = \{z : |z| < 1\}, H(G) = \{f : f \text{ is analytic on the region } G\}.$ 

- 1. Suppose that  $f : \mathbb{D} \to \mathbb{D}$  is holomorphic, with f having a zero of order at least n at z = 0, where  $n \ge 1$ . Show that  $|f^{(n)}(0)| \le n!$ .
- 2. Suppose that  $\mathcal{F} \subset H(\mathbb{D})$ . Let  $A = \{z : \frac{1}{2} < |z| < 1\}$  and define  $\mathcal{F}_A = \{f|_A : f \in \mathcal{F}\}$ . Show that if  $\mathcal{F}_A$  is normal in H(A) then  $\mathcal{F}$  is normal in  $H(\mathbb{D})$ .
- 3. Suppose f is analytic in  $\mathbb{D}$  and  $|f(z)| \to 1$  as  $|z| \to 1^-$ . Show that the number of solutions (counting multiplicity) of  $f(z) = \alpha$  is the same for all  $\alpha \in \mathbb{D}$ .
- 4. Describe the branches of  $z \mapsto z^{\frac{1}{4}}$  on  $G = \mathbb{C} \setminus (-\infty, 0]$ . How many branches are there? Which is the principal branch? What is the range of each branch? Describe the associated Riemann surface.
- 5. Let  $G = \{z \in \mathbb{D} | \text{Im } z > 0\} \setminus \{iy \mid 0 < y \leq \frac{1}{2}\}$ . See the figure below. Give an explicit one-to-one conformal map which maps G onto  $\mathbb{D}$ .



6. Let 
$$f(z) = \frac{1}{z^2(z-1)(z-2)}$$
. Find  $\int_{\gamma} f(z) dz$  where  $\gamma$  is as pictured below:



7. Find the fallacy in the following argument: Let *m* and *n* be two arbitrary integers. Then

$$e^{2m\pi i} = e^{2n\pi i}$$
:

hence,

$$\left(e^{2m\pi i}\right)^i = \left(e^{2n\pi i}\right)^i.$$

It follows that

$$e^{-2m\pi} = e^{-2n\pi}$$

Since  $-2m\pi$  and  $-2n\pi$  are both real,  $-2m\pi = -2n\pi$ . Therefore,

m = n.

8. If u is a positive harmonic function on the ball  $\{z \mid |z - a| < R\}$ , show that

$$\frac{1}{3}u(a) \le u(z) \le 3u(a)$$

for  $|z-a| \leq \frac{R}{2}$ .