Complex Analysis **Prelimary Examation** May 2002

Do all eight problems.

1. State and prove Schwarz's Lemma.
2. Find a conformal, one-to-one map f from $\mathbb{D} = \{z : |z| < 1\}$ onto
$$G = \{w : |\operatorname{Im} w| < \pi/2\} \setminus \{w : w \leq -1\}$$
such that $f(0) = 1$.
3. Evaluate the integral
$$\int_0^{2\pi} \frac{d\theta}{1 + \sin^2\theta} \ .$$
4. Prove the reflection principle: If $H = \{z \in \mathbb{C} : \operatorname{Im}\, z > 0\}$, and if f is a continuous function on $\overline{H}$, analytic on H and if f is real on the real axis, then f can be analytically extended from H to all of $\mathbb{C}$.
5. A function f is said to satisfy the *Lipschitz condition* on $\mathbb{C}$ if there exists a positive constant M such that
$$|f(z_1) - f(z_2)| \leq M \cdot |z_1 - z_2| \qquad \text{for all } z_1, z_2 \in \mathbb{C} \ .$$
Find all entire functions that satisfy the Lipschitz condition on $\mathbb{C}$.
6. Suppose $f(z) = \sum_{n=0}^{\infty} a_n (z-c)^n$ has the property that the series $\sum_{n=0}^{\infty} f^{(n)}(c)$ converges. Show that f is an entire function.
7. Classify [type (and order where applicable)] all of the isolated singularities of the following functions (including any isolated singularities at the point at ∞):
   a) $f(z) = \dfrac{\sin^2 z}{z^4}$
   b) $f(z) = \sin\left(\dfrac{1}{z}\right) + \dfrac{1}{z^2(z-1)}$
   c) $f(z) = \csc z - \dfrac{1}{z}$
8. Let w_1 and w_2 be distinct points in $\mathbb{C}$ and let L be the perpendicular bisector of the line segment connecting them. Describe the image of L under the map
$$F(z) = \frac{z - w_1}{z - w_2} \ .$$