Preliminary Examination 1998
Complex Analysis

Do all problems.

Notation:

$\mathbb{R} = \{ x : x \text{ is a real number} \}$ $\quad$ $\mathbb{C} = \{ z : z \text{ is a complex number} \}$

$B(a,r) = \{ z \in \mathbb{C} : | z - a | < r \}$ $\quad$ $\text{ann}(a,r_1,r_2) = \{ z \in \mathbb{C} : r_1 < | z - a | < r_2 \}$

For $G \subset \mathbb{C}$, let $\mathcal{A}(G)$ denote the set of analytic functions on G (mapping G to $\mathbb{C}$).

1. Let f be an entire function.
   (a) Suppose there exist $a, b \in \mathbb{R}$ such that $|f(z)| \le (a\sqrt{|z|} + b)$ for all $z \in \mathbb{C}$. Show that f is constant.
   (b) Suppose there exist $a, b \in \mathbb{R}$ such that $|f(z)| \le (a|z|^{5/2} + b)$ for all $z \in \mathbb{C}$. What can you say about f?

2. Let f, g be entire functions. Suppose $|f(z)| \le |g(z)|$ for all $z \in \mathbb{C}$. Prove there exists a constant c such that $f \equiv cg$.

3. Let $A(r) = \text{ann}(0,r,1)$, $0 < r < 1$, and $B = B(0,1) \setminus B(¼,¼)$. Show that there exists an r such that $A(r)$ is conformally equivalent to B.

4. Let G_1 and G_2 be simply connected regions, neither region is all of $\mathbb{C}$. Let $a \in G_1$. Suppose that $f, g \in \mathcal{A}(G_1)$ such that f is one-to-one on G_1 with $f(G_1) = G_2$, $g(G_1) \subset G_2$ and $f(a) = g(a)$. Prove $|g'(a)| \le |f'(a)|$.

5. Define the Gamma function, Γ. Prove that $\Gamma(z)\Gamma(1-z) = \pi/\sin(\pi z)$, for z not an integer.

6. Let G be a region in $\mathbb{C}$ and let $\mathcal{F}$ be a subset of $\mathcal{A}(G)$. Prove that if $\mathcal{F}$ is locally bounded, then $\mathcal{F}$ is equicontinuous at each point of G.

7. Let $D_1 = \{ z \in B(0,1) : \text{Im } z > ½\}$. Find a conformal map f which maps D_1 one-to-one and onto $B(0,1)$ such that $f(3i/4) = 0$.

8. Compute $\displaystyle\int_{-\infty}^{\infty} \frac{e^{iax}}{(1+x^2)^2}\,dx$, $a > 0$.

9. Find a non-constant function $f \in \mathcal{A}(B(0,1))$ such that f has infinitely many zeros in $B(0,1)$.

10. Suppose $\alpha \ne 0$ is a root of a polynomial p of degree n with rational coefficients. Prove that $1/\alpha$ is a root of a polynomial of degree at most n with rational coefficients.