Preliminary Examination 1996

Complex Analysis

Do all problems.

Notation:

$\mathbb{R}$ = { real numbers } $\qquad$ $\mathbb{C}$ = { complex numbers}

$B(a,r) = \{ z \in \mathbb{C} : | z - a | < r \}$ $\qquad$ $D = B(0,1)$

$UHP = \{ z \in \mathbb{C} : \text{Im } z > 0 \}$

For $G \subset \mathbb{C}$, let $\mathcal{C}(G)$ denote the set of continuous functions on G (mapping G to $\mathbb{C}$), $\mathcal{A}(G)$ the set of analytic functions on G (mapping G to $\mathbb{C}$), and $\mathcal{H}ar(G)$ the set of harmonic functions on G (mapping G to $\mathbb{R}$).

1. Evaluate the integral $\displaystyle\int_0^\infty \frac{1 - \cos ax}{x^2}\, dx$ for $a \in \mathbb{R}$.

2. Find a one-to-one conformal map of $UHP \setminus B(½,½)$ onto UHP.

3. (a) Prove or disprove: Let $f \in \mathcal{A}(\overline{D})$ be such that $f(\partial D) \subset \mathbb{R}$. Then, f is constant.

   (b) Prove or disprove: Let $f \in \mathcal{A}(\mathbb{C} \setminus \{1\})$ be such that $f(\partial D \setminus \{1\}) \subset \mathbb{R}$. Then, f is constant.

4. (a) Let G be a region. Let $f \in \mathcal{A}(G)$, $f \neq 0$, and let n a positive integer. Assume that f has an analytic n^{th}-root on G, that is, there exists a $g \in \mathcal{A}(G)$ such that $g^n = f$. Prove that f has exactly n analytic n^{th}-roots in G.

   (b) Give an example of a continuous real-valued function on [0,1] that has more than two continuous square roots on [0,1].

5. State the Riemann Mapping Theorem. Prove the uniqueness assertion in the statement of the Riemann Mapping Theorem.

6. Let $u \in \mathcal{H}ar(\mathbb{C})$ be such that $u(z) \leq a \,|\, \log |z| \,| + b$ for some positive constants a and b. Prove that u is constant.

7. Let f be an analytic function such that $f(z) = 1 - z^2 + z^4 - z^6 + \cdots$ for $|z| < 1$. Define a sequence of real numbers $\{a_n\}$ by $f(z) = \sum_{n=0}^{\infty} a_n (z-3)^n$. What is the radius of convergence of the series $\sum_{n=0}^{\infty} a_n z^n$.

8. Let $\cot(\pi z) = \sum_{n=-\infty}^{\infty} a_n z^n$ be the Laurent expansion for $\cot(\pi z)$ on the annulus $1 < |z| < 2$. Compute the coefficients a_n for n = -1, -2, -3, $\cdots$. (Hint: Recall that the only singularities of $\cot(\pi z)$ are simple poles at each of the integers and that the residue at each such singularity is precisely $1/\pi$.)

9. Compute the integral $\int_{|z|=1} (e^{2\pi z} + 1)^{-2}\, dz$.

10. Determine all $f \in \mathcal{A}(D)$ which satisfy $f''(\frac{1}{n}) + f(\frac{1}{n}) = 0$ for $n = 2, 3, 4, \cdots$. Justify your answer.