Preliminary Examination 1995

Complex Analysis

Do all 8 problems.

Notation:

For $G \subset \mathbb{C}$, let $\mathcal{C}(G)$ denote the set of continuous functions on G (mapping G to $\mathbb{C}$) and $\mathcal{H}(G)$ the set of analytic functions on G (mapping G to $\mathbb{C}$).

1. What are all possible values of $\displaystyle\int_\gamma \frac{dz}{1+z^2}$ where γ may be any rectifiable path in $\mathbb{C}$ from 0 to 1 ?

2. Prove that $\displaystyle f(z) = \int_1^\infty \frac{x^z - 1}{e^x - 1}\,dx$ is analytic on $\operatorname{Re} z > 1$.

3. Let $D = \mathbb{C} \setminus [0,1]$. Show that if $f \in \mathcal{H}(D) \cap \mathcal{C}(\overline{D})$, then f is entire.

4. Let f be analytic on $|z| < 1$. Show that if $|f(z)| < 1$ for $|z| < 1$, then there exists a unique root in $|z| < 1$ for which $f(z) = z$. Determine whether the above hypothesis can be weakened to $|f(z)| \leq 1$ for $|z| < 1$ and still imply the conclusion that there exists a unique root in $|z| < 1$ for which $f(z) = z$?

5. Let f be analytic on $|z| < R$. For $0 < r < R$ let $\displaystyle A(r) = \min_{|z|=r} \operatorname{Re} f(z)$. Show that $A(r)$ is a strictly decreasing function (of r) unless f is constant.

6. Given any rational function f with no poles on $|z| = 1$, show there exists a rational function g with no poles on $|z| < 1$ such that $|f(z)| = |g(z)|$ on $|z| = 1$.

7. Let $D = \{ z : |z| < 1 \}$. Let $l = \{ iy : y \geq 1 \} \cup \{ iy : y \leq -1 \}$ and $E = \mathbb{C} \setminus l$. Find a one-to-one, conformal mapping f from D onto E such that the power series expansion for f at 0 has real coefficients.

8. For g $\in \mathcal{C}(\mathbb{C})$, let $Z(g)$ denote the zero set of g. Suppose that $f_n \in \mathcal{H}(\mathbb{C})$ and that $Z(f_n) \subset \mathbb{R}$, n = 1, 2, 3, $\cdots$. Suppose that $f_n \to f$ locally uniformly on $\mathbb{C}$. Show that $f \in \mathcal{H}(\mathbb{C})$ and that $Z(f) \subset \mathbb{R}$.