Preliminary Examination
Complex Analysis
May 1994

Provide complete solutions for each problem.

1. Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{e^{x/\pi}}{1+e^{x}}\,dx$. Carefully justify any limit passages you make.

2. Let $D = \{ z : |z| < 1\}$. Let $\mathcal{F} = \{ f : f$ is analytic on D, $f(z) = a_1 z + a_2 z^2 + a_3 z^3 + \cdots$ with $|a_n| \le n$ for $n \in \mathbb{Z}^+\}$. Prove that $\mathcal{F}$ is normal.

3. Let f be analytic on $\overline{B(0,r)}$ with $f(0) \neq 0$. Suppose $\{ a_1, a_2, \ldots, a_n \}$ are the zeros of f on $B(0,r)$, counted according to multiplicity.

   a) Show that if $|\zeta| = r$, then

$$\frac{1}{2\pi}\int_0^{2\pi} \log|re^{i\theta} - \zeta| = \log r$$

   b) Show that the geometric mean of f on $|z| = r$ is $|f(0)|\dfrac{r^n}{\prod |a_k|}$, i.e., show

$$e^{\frac{1}{2\pi}\int_0^{2\pi}\log|f(re^{i\theta})|\,d\theta} = |f(0)|\frac{r^n}{\prod |a_k|}.$$

   Hint: Construct a function F which has no zeros on $B(0,r)$ and for which $|F(z)| = |f(z)|$ on $|z| = r$. Consider the two cases: (i) F has no zeros on $|z| = r$; (ii) F has zeros on $|z| = r$ [use part a)].

4. Let Γ be the square given by

$$\Gamma = \{ z : -5 \le \mathcal{R}e\, z, \mathcal{I}m\, z \le 5\}.$$

   Suppose that f is analytic on a domain containing Γ and $f''(1) = 1$. Define $M = \max_{z\in\Gamma} |f(z)|$. Find an example of a function f with $M = 5$ or prove that none exists.

5. Prove that the equation

$$z + 3 + 2e^{z} = 0$$

   has precisely one root in the left-half plane.

6. Suppose that f is analytic and nowhere zero in $\Omega = \{ z : \mathcal{Re}\, z < 1\}$. Prove that $\log |f|$ is harmonic in Ω.

7. Let S be a domain in the complex plane. Assume that B is a ball of positive radius contained in S. Let f be an analytic function on S. Prove that if $\mathcal{Im}\, f$ is constant on B, then f is constant on S.

8. For $n \in Z^+$ let

$$P_n(z) = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots + \frac{z^n}{n!}.$$

Prove that for each $r > 0$ there exists an integer $N = N(r)$ such that for all $n > N$ all of the roots of P_n lie outside of the circle $\{ z : |z| = r \}$.

9. Find the linear fractional transformation which maps the points 2, $1+i$, 0 to the points 0, 1, ∞. Describe the image of the exterior of the union of the disks $\{z : |z - 1| < 1\}$ and $\{z : |z + 1| < 1\}$.