# Preliminary Examination
# Complex Analysis
# May 1994

**Provide complete solutions for each problem.**

1. Evaluate $\int_{-\infty}^{\infty} \frac{e^{x/\pi}}{1 + e^x}\, dx$. Carefully justify any limit passages you make.

2. Let $D = \{ z : |z| < 1\}$. Let $\mathcal{F} = \{ f : f$ is analytic on $D$, $f(z) = a_1 z + a_2 z^2 + a_3 z^3 + \cdots$ with $|a_n| \le n$ for $n \in \mathbb{Z}^+\}$. Prove that $\mathcal{F}$ is normal.

3. Let $f$ be analytic on $\overline{B(0,r)}$ with $f(0) \neq 0$. Suppose $\{ a_1, a_2, \ldots, a_n \}$ are the zeros of $f$ on $B(0,r)$, counted according to multiplicity.

   a) Show that if $|\zeta| = r$, then

   $$\frac{1}{2\pi} \int_0^{2\pi} \log |re^{i\theta} - \zeta| = \log r$$

   b) Show that the geometric mean of $f$ on $|z| = r$ is $|f(0)| \frac{r^n}{\prod |a_k|}$, i.e., show

   $$e^{\frac{1}{2\pi}\int_0^{2\pi} \log |f(re^{i\theta})|\, d\theta} = |f(0)| \frac{r^n}{\prod |a_k|}.$$

   Hint: Construct a function $F$ which has no zeros on $B(0,r)$ and for which $|F(z)| = |f(z)|$ on $|z| = r$. Consider the two cases: (i) $F$ has no zeros on $|z| = r$; (ii) $F$ has zeros on $|z| = r$ [ use part a)].

4. Let $\Gamma$ be the square given by

   $$\Gamma = \{ z : -5 \le \mathcal{Re}\, z, \mathcal{Im}\, z \le 5\}.$$

   Suppose that $f$ is analytic on a domain containing $\Gamma$ and $f''(1) = 1$. Define $M = \max_{z \in \Gamma} |f(z)|$. Find an example of a function $f$ with $M = 5$ or prove that none exists.

5. Prove that the equation

   $$z + 3 + 2e^z = 0$$

   has precisely one root in the left-half plane.

6. Suppose that $f$ is analytic and nowhere zero in $\Omega = \{ z : \mathcal{Re}\, z < 1\}$. Prove that $\log |f|$ is harmonic in $\Omega$.

7. Let $S$ be a domain in the complex plane. Assume that $B$ is a ball of positive radius contained in $S$. Let $f$ be an analytic function on $S$. Prove that if $\mathcal{Im}\, f$ is constant on $B$, then $f$ is constant on $S$.

8. For $n \in Z^+$ let

$$P_n(z) = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots + \frac{z^n}{n!}.$$

Prove that for each $r > 0$ there exists an integer $N = N(r)$ such that for all $n > N$ all of the roots of $P_n$ lie outside of the circle $\{ z : |z| = r \}$.

9. Find the linear fractional transformation which maps the points 2, $1+i$, 0 to the points 0, 1, $\infty$. Describe the image of the exterior of the union of the disks $\{z : |z - 1| < 1\}$ and $\{z : |z + 1| < 1\}$.