Preliminary Exam
Complex Analysis
May 1993

Do all problems. In each case, justify your answer.

(1) Let $f(z) = z^i$, where the branch chosen is the principal branch. Show that there is a constant $C \geq 0$ such that $|f(z)| \leq C$ for all z in the domain of definition of f.

(2) Let $f(z)$ be an entire function such that $|f(z)| \leq 1 + |z|^{3/2}$. Show that $f(z) = az + b$ for some constants a and b.

(3) Suppose that f is analytic on a neighborhood of the closed unit disk $\overline{B}(0;1)$ and that f is not zero at any point of $\partial\overline{B}(0;1)$. Show that

$$\max_{|z|=1} \operatorname{Re}\left(\frac{z f'(z)}{f(z)}\right)$$

is greater than or equal to the number of zeros of f in $\overline{B}(0;1)$.

(4) State and prove Morera's Theorem.

(5) Use the method of residues to to compute the integral

$$\int_0^\infty \frac{1}{1+x^3}\,dx.$$

Justify the steps in your argument.

(6) Let A be the annulus $A = \{ z \in \mathbb{C} : 1 < |z| < 3 \}$. Show that the function $f(z) = 1/z^2$ can **not** be uniformly approximated by polynomials on A.

(7) Let $P = \{ z \in \mathbb{C} : \operatorname{Im}(z) > 0 \}$ be the upper half-plane. Let F be the family of functions

$$F = \{ f \mid f : P \to P,\ f \text{ analytic},\ f(i) = i \}.$$

Find the value of

$$\max_{f \in F} |f'(i)|.$$

(8) Let G be the region defined by

$$G = \{ z \in \mathbb{C} : \operatorname{Re}(z) > 0 \} \cap \{ z \in \mathbb{C} : |z| > 1 \}$$

Find a one-to-one analytic mapping of G onto the unit disk.