Complex Analysis Preliminary Exam 1992

In all problems *prove* your answer to be correct.

1. Give an example of a function $f : \mathbb{C} \to \mathbb{C}$ for which $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist and are continuous mappings from $\mathbb{C}$ to $\mathbb{C}$, but for which $\lim_{z \to 0} \frac{f(z) - f(0)}{z}$ fails to exist.

2. a) Suppose $f : \mathbb{R} \to \mathbb{R}$ is differentiable. Must if follow that $f' : \mathbb{R} \to \mathbb{R}$ is differentiable?

   b) Suppose $f : \mathbb{C} \to \mathbb{C}$ is analytic (holomorphic). Must it follow that $f' : \mathbb{C} \to \mathbb{C}$ is analytic (holomorphic)?

3. a) Suppose $f : \mathbb{R} \to \mathbb{R}$ is infinitely differentiable and for every non-negative integer n, $f^{(n)}(0) = 0$. Must it follow that $f \equiv 0$ on some neighborhood of 0?

   b) Suppose $f : \mathbb{C} \to \mathbb{C}$ is analytic (holomorphic) and for every non-negative integer n, $f^{(n)}(0) = 0$. Must it follow that $f \equiv 0$ on some neighborhood of 0? Must it follow that $f \equiv 0$ on all of $\mathbb{C}$?

4. Let $D = \{ z : |z| < 1 \}$, $\hat{D} = D \setminus \{ (-1, -\frac{1}{2}] \cup [\frac{1}{2}, 1) \}$, and $f : \hat{D} \to D$ with $f(0) = 0$, f conformal, one-to-one and onto.
   a) Find a formula for f.
   b) Show $|f'(0)| > 0$.

5. Show that the punctured unit disk $D^* = \{ z : 0 < |z| < 1 \}$ is *homeomorphic* to $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$, but is not *conformally equivalent* to $\mathbb{C}^*$.

6. Let a be a fixed real number. Find the set of z in $\mathbb{C}$ for which $\sum_{n=1}^{\infty} n^{-i(z^2 + a)}$ is convergent to an analytic function.
7. Find the number of real roots and the number of non real roots of the equation 3 tan z - z = 0. (Hint, consider the closed curve defined by the square with corners at $N\pi(\pm 1 \pm i)$ for integer values N.)
8. State and prove Montel's theorem.