Preliminary Examination
Complex
May 1991

Attempt *all* problems. Be as *brief* and *neat* as possible. *Partial credit* for *correct*, but incomplete results will be given. We do not expect every student to be able to completely solve every problem. We do expect to see some problems completely solved and reasonable progress on the others.

1. State and prove Rouche's theorem.

2. List 5 properties which hold for analytic (holomorphic) functions on $D = \{ |z| < 1 \}$, which need not hold for functions which are only Frechet (real) differentiable on D. In each case provide and example of a Frechet differentiable function on D which fails to have the stated property. (For your purposes Frechet differentiable means $\partial/\partial x$ and $\partial/\partial y$ exist and are continuous.)

3. a) Suppose f is a non-constant analytic function on the punctured disk $D^* = \{ z : 0 < |z| < 1 \}$. Further, suppose $f(1/n) = 0$ for every integer $n > 1$. Determine all the possible types of singularities f can have at 0.
   b) Provide an example of a function satisfying part a).

4. a) Find a conformal map from the disk $D = \{ |z| < 1 \}$ to the annulus $A = \{ 1 < |z| < e \}$.
   b) Does there exist a one-to-one conformal map satisfying part a)? Justify your answer.

5. For $D = \{ |z| < 1 \}$ let

$$\begin{aligned} C(D) &= \{ f : D \to \mathbb{C} \mid f \text{ is continuous} \} \\ C^\infty(D) &= \{ f : D \to \mathbb{C} \mid \partial^{n+m} f / \partial x^n \partial y^m \text{ exist and} \\ &\qquad \text{are continuous } \forall \text{ integers } n,m >= 0 \} \\ C^\omega(D) &= \{ f : D \to \mathbb{C} \mid f \text{ is a convergent power series} \\ &\qquad \text{in the real variables } x \text{ and } y \} \\ H(D) &= \{ f : D \to \mathbb{C} \mid f \text{ is analytic} \} \end{aligned}$$

   Prove $H(D) \not\subset C^\omega(D) \not\subset C^\infty(D) \not\subset C(D)$.

   (You need not prove the theorems you use to obtain the inclusions, but please provide explicit examples to show each inclusion is proper!)

6. a) Show $\sum_{n=1}^{\infty} e^{-n} \sin nz$ defines an analytic function on $\{ -1 < Im\, z < 1 \}$.
   b) Find a closed form for the function in part a).

7. a) Suppose f is meromorphic on a neighborhood of $\{Im\, z > 0\}$ and has no poles on the real axis.

   Also, suppose for $|z|$ sufficiently large $|f(z)| \le K/|z|^p$, where $p > 1$ and K are constants.

   Prove $\int_{-\infty}^{\infty} f(x)\, dx = 2\pi i \sum_{n=1}^{k} \text{Res}(f; z_n)$, where z_n denote the poles of f in $\{Im\, z > 0\}$.

   b) Compute $\int_0^{\infty} \frac{x^2}{1 + x^4}\, dx$.

8. a) Suppose f is a rational function, all the zeros and poles of which are of even order. Prove there exists a rational function g such that $g^2 = f$.

   b) Suppose f is meromorphic on a simply connected set G and has all its zeros and poles of are of even order. Prove there exists a meromorphic function g on G such that $g^2 = f$.