Preliminary Examination
Complex Analysis
May 1990

Work all problems.

1. Construct an analytic, one-to-one map of $\{ z : |z| < 1 \}$ onto $\{ w : |Im\, w| < 3\pi/4\} \setminus (-\infty,0]$.

2. Evaluate $\int_0^\infty \frac{x^{\frac{1}{2}}}{x^2 + 3x + 2}\, dx$ and justify your solution.

3. Let f be an entire function. Suppose f is real on the real axis and pure imaginary on the imaginary axis. Prove that f is an odd function.

4. Show that the only periodic rational functions on the complex plane are the constant functions.

5. a) Define what it means for a family of analytic functions to be locally bounded on a region.

   b) Prove that a family F of analytic functions on a region G in the plane is compact if and only if it is closed and locally bounded.

Suppose $f : G \to \mathbb{C}$ is analytic and the closed unit ball, $\overline{B(0,1)}$, is contained in the open set G. The Cauchy integral formula tells how to obtain the value of f at points in $B(0,1)$ from the values of f on the boundary of $\overline{B(0,1)}$. Use the Cauchy integral formula to prove the following two facts.

6. $f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n$ for all z in $B(0,1)$.

7. There exists an analytic function $F : B(0,1) \to \mathbb{C}$ such that $F'(z) = f'(z)$ for all z in $B(0,1)$.