A Short Proof of a Conjecture on the Integral Means of the Derivative of a Convex Function

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For $d > 0$ let $D_d = \{z : |z| < d\}$ with $D_1 = D$ and let ∂D_d denote the boundary of D_d. Let S be the standard class of analytic, univalent functions f on D, normalized by $f(0) = 0$ and $f'(0) = 1$ and let K denote the well-known class of convex functions in S. For $0 \le \alpha < 1$ let $S^*(\alpha)$ denote the subclass of S of starlike functions of order α, i.e., a function $f \in S^*(\alpha)$ if and only if f satisfies the condition $Re\, zf'(z)/f(z) > \alpha$ on D. It is well known that $K \subset S^*(1/2)$.

For $F \subset S$ and for $1/4 \le d \le 1$ let

$$F_d = \{f \in F : \min_{z\in D} \left|\frac{f(z)}{z}\right| = d\}.$$

Note, that $K_d = \emptyset$ for $1/4 \le d < 1/2$.

A general problem which arose out the authors' work in the early 80's on omitted value problems for convex functions, see [Ba] and [BaPe1], is the following: Given $F \subset S$ and $1/4 \le d \le 1$ determine the sharp constant $A = A(F_d)$ such that for any $f \in F_d$

$$I_{-1}(f') = \frac{1}{2\pi}\int_0^{2\pi} \left|\frac{1}{f'(e^{i\theta})}\right| d\theta \le \frac{A}{d}. \tag{1}$$

Further, determine the sharp constant $A = A(F) = \sup_d A(F_d)$.

It follows fairly easily from subordination theory that $A(S^*(1/2)) \le 4/\pi$. Furthermore, this estimate is sharp for $S^*(1/2)$ since the functions $f_n(z) = z/(1-z^n)^{1/n}$ belong to $S^*(1/2)$ for each $n > 0$. However, this estimate is not sharp for the class K of convex functions which is a proper subset of

*Research supported in part by Texas Advanced Research Program grant #003644-125.

$S^*(1/2)$. Considerable numerical evidence suggested to the authors to make the following conjecture.

Conjecture. For each d, $1/2 \leq d \leq 1$, $A = A(K_d) = 1$ in (1) with equality holding for all domains which are bounded by regular polygons centered at the origin.

This conjecture was announced in March 1985 at the Symposium on the Occasion of the Proof of the Bieberbach Conjecture at Purdue University. It also appeared as Conjecture 8 in the first author's "Open Problems and Conjectures in Complex Analysis" in [Ba]. It was thought, by many function theorists, that the conjecture would be easily settled, given the vast literature on convex functions and the large research base for determining integral mean estimates, see [Du].

An initial difficulty was the non-applicability of Baerenstein's circular symmetrization methods, since convexity, unlike univalence and starlikeness, is not preserved under circular symmetrization. Although Steiner symmetrization does preserve convexity, see [Va], it did not appear to be helpful for the problem and, indeed, we found that extremal domains need possess no standard symmetry.

A confusing issue, which also arises, is that the integral means of the standard approximating functions f_n in K defined by

$$f_n'(z) = \prod_{k=1}^{n}(1 - ze^{i\theta_k})^{-2\alpha_k}, 0 < \alpha_k \leq 1, \sum_{k=1}^{n} \alpha_k = 1$$

decrease when the arbitrarily distributed θ_k are replaced by uniformly distributed $t_k = k\pi/n$, as was shown in [Zh]. The conjecture suggests that multiplication by the minimum modulus d must overcome this decrease.

We make the following definition.

Definition. Let Γ be a curve in $\mathbb{C}$ such that the left- and right-hand tangents to the curve Γ exist at each point on Γ. The curve Γ will be said to *circumscribe a circle* C if the left- and right-hand tangents to the curve Γ at each point on Γ lie on tangent lines to the circle C.

We will employ the following notation.

Notation. Let $f \in S$ and suppose that γ is a subarc of ∂D on which f is smooth. For $z = e^{i\theta} \in \gamma$ let $d_\theta =< f(z), \frac{zf'(z)}{|zf'(z)|} >$, i.e, d_θ is the directed length of the projection of $f(z)$ onto the outward unit normal to $\partial f(D)$ at $f(z)$.

In 1993 [BaPe2], we proved the following theorem which verified the conjecture.

Theorem A. Let $f \in K$, $d = \min_{\theta} |f(e^{i\theta})|$ and $d^* = \sup_{\theta} d_\theta$. Then,

$$\frac{1}{d^*} \leq \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{|f'(e^{i\theta})|} d\theta \leq \frac{1}{d} \tag{2}$$

with equality holding if $\partial f(D)$ circumscribes ∂D_d.

The original proof, which was lengthy, was based on the Julia variational formula and depended heavily on the convexity of f. We obtained, arising out of the proof, the rather unexpected sufficient condition for equality to occur in (2) for the classes K_d. However, because the proof used a scheme to approximate convex functions by polygonally convex functions, we did not obtain a necessary condition for equality.

We have devised a new, simpler proof for the conjecture which extends Theorem A. The proof releases the convexity requirement and validates the necessity of the sufficient condition.

Theorem B. Let $f \in S^*(\alpha)$ for some $\alpha > 0$. Suppose f is smooth on $X \subset \partial D$ where X is a countable union of pairwise disjoint subarcs of ∂D such that the complement of X in ∂D has measure zero. Let $d_* = \inf_{\theta \in X} d_\theta$, $d^* = \sup_{\theta \in X} d_\theta$. Then,

$$\frac{1}{d^*} \leq \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{|f'(e^{i\theta})|} d\theta \leq \frac{1}{d_*} \tag{3}$$

with equality holding if and only if $\partial f(D)$ circumscribes ∂D_{d_*}.

Remark. In general $d_* \leq d = \min_{z \in D} |\frac{f(z)}{z}|$ with equality if $f \in K$.

Proof. Let $X = \bigcup_{k=1}^{\infty} \gamma_k$, where each γ_k is a subarc of ∂D. We have from the Cauchy Integral Formula, with $z = e^{i\theta}$, that

$$\begin{aligned}
1 &= \lim_{r \to 1} \frac{1}{2\pi} \int_0^{2\pi} \frac{f(rz)}{rzf'(rz)} d\theta \\
&= \frac{1}{2\pi} \int_0^{2\pi} Re \frac{f(z)}{zf'(z)} d\theta \\
&= \frac{1}{2\pi} \int_0^{2\pi} \frac{|f(z)| \cos(\arg \frac{f(z)}{zf'(z)})}{|zf'(z)|} d\theta \qquad (4) \\
&= \frac{1}{2\pi} \sum_{k=1}^{\infty} \int_{\gamma_k} \frac{< f(z),\ \frac{zf'(z)}{|zf'(z)|} >}{|f'(z)|} d\theta
\end{aligned}$$

$$= \frac{1}{2\pi} \sum_{k=1}^{\infty} \int_{\gamma_k} \frac{d_\theta}{|f'(z)|} d\theta$$

The restriction of f to $S^*(\alpha)$ with $\alpha > 0$ assures the passage of the limit in (4). Replacing d_θ by d^* and d_* in this last integral gives the left- and right-hand inequalities in (3), respectively. Equality clearly occurs if $\partial f(D)$ circumscribes ∂D_{d_*}, for in this case $d_* = d_\theta = d^*$ on X. Conversely, if $\partial f(D)$ does not circumscribe ∂D_{d_*}, then there must exist a subarc I, which must lie in one of the γ_k, of positive (linear) measure on which

$$d_* < d_\theta < d^*. \tag{5}$$

The strictness of (5) on I implies that strict inequality holds in both the left- and right-hand inequalities in (3). ■

The authors would like to thank Al Baerenstein for his helpful suggestion which led to their shorten proof.

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