Sharp bounds on the H_p means of the derivative of a convex function for $p = -1$.

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Dedicated to the memory of our friend Glenn Schober.

1 Introduction

For $d > 0$ let $D_d = \{z : |z| < d\}$ with $D_1 = D$ and let ∂D_d denote the boundary of D_d. Let S be the standard class of analytic, univalent functions f on D, normalized by $f(0) = 0$ and $f'(0) = 1$. Let K denote the well known class of convex functions in S and let

$$K_d = \{f \in K : \min_{z \in D} \left| \frac{f(z)}{z} \right| = d\}. \tag{1}$$

We will frequently (verbally) identify an analytic, univalent function f on D with normalization

$$f(0) = 0 \text{ and } f'(0) > 0 \tag{$*$}$$

with its range $f(D)$ and conversely, since the Riemann mapping theorem guarantees that we can do so without ambiguity. Specifically, we will refer to convex domains which contain the origin and mean the convex functions f which map onto those domains and satisfy $(*)$.

A problem which arose out the authors' work in the early 80's on omitted value problems for univalent functions, see [1] and [3], is the following: Given

$d, \frac{1}{2}, \leq d \leq 1$, determine a sharp constant $A = A(d)$ such that for any $f \in K_d$

$$I_{-1}(f') = \frac{1}{2\pi} \int_0^{2\pi} \left| \frac{1}{f'(e^{i\theta})} \right| d\theta \leq \frac{A}{d}. \tag{2}$$

It follows fairly easily from subordination theory that $A \leq 4/\pi$, but this is not sharp for convex functions. For $0 \leq \alpha < 1$ let $S^*(\alpha)$ denote the usual subclass of S of starlike functions of order α, i.e., a function $f \in S^*(\alpha)$ if and only if f satisfies Re $zf'(z)/f(z) > \alpha$ on D, see [9]. We will show that $4/\pi$ is sharp for the class $S^*(\frac{1}{2})$ which strictly contains K. In Theorem 3 we determine for each $\alpha, 0 \leq \alpha < 1, A = A(\alpha)$ in (2) for the class $S^*(\alpha)$ and show that it is sharp.

Considerable numerical evidence suggested to the authors to make the following conjecture:

Conjecture 1 *For each $d, \frac{1}{2} \leq d \leq 1, A = A(d) = 1$ in (2) for the class K_d with equality holding for all domains which are bounded by regular polygons centered at the origin.*

This conjecture was announced at several talks and conferences over the last decade including the first author's talk on "Open problems in complex analysis" given at the Symposium on the Occasion of the Proof of the Bieberbach Conjecture at Purdue University in March 1985. It also appeared as Conjecture 8 in the first author's "Open Problems and Conjectures in Complex Analysis" in [1]. It was thought, by many function theorists, that the conjecture would be easily settled, given the vast literature on convex functions and the large research base for determining integral mean estimates, see [6].

An initial difficulty was the non-applicability of Baernstein's general circular symmetrization methods. The non-convexity of the circularly symmetrized square shows that convexity, unlike univalence and starlikeness, is not preserved under circular symmetrization. Although Steiner symmetrization does preserve convexity, see [10], it did not appear to be helpful for the problem and, indeed, we will show that an extremal domain need posses no standard symmetry.

A confusing issue, which also arises, is that the integral means $I_{-1}(f'_n)$ for the standard approximating functions f_n in K, which map D onto n-sided convex polygons and which are defined by $f'_n(z) = \prod_{k=1}^{n} (1 - ze^{i\theta_k})^{-2\alpha_k}$,

$0 < \alpha \leq 1, \sum_{k=1}^{n} \alpha_k = 1$, decrease, as was recently shown in [13], when the arbitrarily distributed θ_k are replaced by uniformly distributed $t_k = k\pi/n$. The regular polygons, produced by the uniformly distributed t_k, are the conjectured extremal domains. The conjecture suggests that multiplication by the minimum modulus d must overcome this decrease.

We shall verify the conjecture in Theorem 1. We also obtain, arising out of the proof, a rather unexpected sufficient condition for equality to occur in (2) for the classes K_d. Additionally, modifying the proof of Theorem 1, we obtain in Theorem 1$*$ and its corollary sharp upper and lower bounds for the integrals means $I_{-1}(f')$.

2 Statements of the Main Theorems

Let K_d be defined as in (1). We will say that a convex curve Γ *circumscribes* a circle C if the left- and right-hand tangents at each point of Γ are tangent to C.

Theorem 1 *Let d be given, $\frac{1}{2} \leq d \leq 1$, and let $f \in K_d$. Then*

$$\frac{1}{2\pi} \int_0^{2\pi} \left| \frac{1}{f'(e^{i\theta})} \right| d\theta \leq \frac{1}{d} \tag{3}$$

with equality holding in (3) *if the boundary of $f(D)$ circumscribes ∂D_d.*

Using Theorem 1 and earlier work of the first author we also prove the following result, which gives a bound for the convex case of Brennan's conjecture [5] for arbitrary univalent functions. Let d be given, $\frac{1}{2} \leq d \leq 1$, and let F_d be the uniquely defined function in K_d which maps D onto the convex domain having as its boundary an arc, C_d, on ∂D_d which is symmetric about the positive real axis and also has as its boundary two lines, tangent to ∂D_d at the endpoints of C_d. Note that $F_1(z) = z$ and $F_{\frac{1}{2}}(z) = z/(1+z)$. Note, also, the $\partial F_d(D)$ circumscribes ∂D_d.

Theorem 2 *Let d be given, $\frac{1}{2} \leq d \leq 1$, and let $f \in K_d$. Then*

$$\iint_D |f'(z)|^{-1} dx dy \leq 2\pi \int_0^1 \frac{r^2 dr}{F_d(r)}. \tag{4}$$

Finally, we shall prove

Theorem 3 *Let α be given, $0 \le \alpha < 1$. If $f \in S^*(\alpha)$ and $\min_{z\in D} |f(z)/z| = d$, then*

$$\frac{1}{2\pi}\int_0^{2\pi}\left|\frac{1}{f'(e^{i\theta})}\right| d\theta \le \frac{A(\alpha)}{d} \tag{5}$$

where

$$A(\alpha) = \frac{4}{\pi}\frac{\arctan\sqrt{2\alpha-1}}{\sqrt{2\alpha-1}} \tag{6}$$

and $A(\alpha)$ is sharp for the class $S^*(\alpha)$.

3 Proofs of Main Theorems

Proof of Theorem 1.

Let d be given, $\frac{1}{2} \le d \le 1$, and let $f \in K_d$. The major idea in proving Theorem 1 is to produce, using the convexity of the domain $\Omega = f(D)$, two varied domains Ω^* and Ω^{**} which will satisfy the inequality

$$\Omega^* \subseteq \Omega^{**}. \tag{7}$$

For any simply connected domain Λ which contains the origin let m.r.(Λ) denote the mapping radius of Λ. Recall that if $\Lambda = g(D)$, where g is analytic and univalent on D and g satisfies the normalization $(*)$, then m.r.$(\Lambda) = g'(0)$. Let Λ^+ be a varied domain of Λ which contains Λ. Denote the change in the mapping radius from the domain Λ to Λ^+ by Δm.r.(Λ^+, Λ).

The domain containment in (7) and subordination will imply that

$$\Delta\text{m.r.}(\Omega^*, \Omega) \le \Delta\text{m.r.}(\Omega^{**}, \Omega) \tag{8}$$

from which the conclusion of Theorem 1 will be obtained.

The varied domain Ω^{**} is constructed as follows: Let $\epsilon > 0$ be given sufficiently small. Define f_ϵ^{**} by $f_\epsilon^{**}(z) = (1+\epsilon)f(z)$ and set $\Omega^{**} = f_\epsilon^{**}(D)$. This has the effect of radially projecting each point $\omega \in \partial f(D)$ to the point $\omega^{**} = (1+\epsilon)\omega$. This gives that

$$[f_\epsilon^{**}(z)]'|_{z=0} = \text{m.r.}(\Omega^{**}) = 1+\epsilon \tag{9}$$

.

$$\Delta\text{m.r.}(\Omega^{**}, \Omega) = \epsilon$$

To construct Ω^* we project each point $\omega \in \partial\Omega$ outward in the direction of the normal, where it exists, a distance ϵd. Let the extension of f to ∂D also be denoted by f. It is well known that this extension maps ∂D onto a Jordan curve (on the Riemann sphere if f is unbounded), $\Gamma = f(\partial D)$, having one-sided tangents everywhere with the set of points having different one-sided tangents being at most countable, see [8].

We define the curve Γ^*, which will be the boundary of Ω^*, as follows: At each point where it exists, let $n(\omega)$ be the unit outward normal to Γ at $\omega = f(e^{i\theta})$. Since $n(\omega) = n[f(e^{i\theta})] = e^{i\theta} f'(e^{i\theta})/|f'(e^{i\theta})|$, we will also define at each finite point where the left- and right-hand tangents differ two limiting normal vectors as $n^1[f(e^{i\theta})] = n[f(e^{i(\theta-0)})]$ and $n^2[f(e^{i\theta})] = n[f(e^{i(\theta+0)})]$. We associate with each point $\omega \in \Gamma$ the point $\omega^* = \omega + \epsilon d[n(\omega)]$ or, where appropriate, the limiting points $\omega^{*j} = \omega + \epsilon d[n^j(\omega)], j = 1$ and 2. To complete Γ^* we extend the tangent line to Γ^* at each ω^{*j}. Each such tangent line to Γ^* at ω^{*j} is parallel to a one-sided tangent line to Γ at ω, so that the resulting curve Γ^* bounds a convex domain Ω^*. We let f_ϵ^* be the function which maps D onto Ω^* normalized by $(*)$.

To determine the change in the mapping radius from Ω to Ω^* we apply the Hadamard variational formula as developed in [2] and earlier in [12]. If Γ is bounded, then we can apply the Hadamard variational formula, cast in the form of the Julia variation, to obtain

$$\Delta\text{m.r.}(\Omega^*, \Omega) = \frac{1}{2\pi} \int_0^{2\pi} \frac{\epsilon d}{|f'(e^{i\theta})|} d\theta + o(\epsilon). \tag{10}$$

However, if Γ is unbounded, we need to show that we can obtain (10) as a limiting argument. First, we define the varied domain Ω_0^* by identifying its boundary Γ_0^*. As in the definition of Γ^*, we associate with each point $\omega \in \Gamma$ the point $\omega^* = \omega + \epsilon d[n(\omega)]$ or, where appropriate, the limiting points $\omega^{*j} = \omega + \epsilon d[n^j(\omega)], j = 1$ and 2. To complete Γ_0^* we connect ω_1^* to ω_2^* by the positively oriented circular arc of radius ϵ and center ω. We identify Ω_0^* as the interior of Γ_0^* and note that we have $\Omega \subsetneq \Omega_0^* \subsetneq \Omega^*$.

Let $0 < r < 1$ and define f_r by $f_r(z) = f((1-r)z)$. Then, for each r we have $f_r(D)$ is bounded. Since f_r is analytic on ∂D, there is for each $\omega_r = f_r(e^{i\theta}) \in f_r(\partial D) = \Gamma_r$ a well-defined outward normal, say, $n(\omega_r)$. Let $\omega_r^* = \omega_r + \epsilon d[n(\omega_r)]$. Let Ω_r^* denote the interior of the curve $\Gamma_r^* = \cup_{\omega_r \epsilon \Gamma_r} \omega_r^*$.

Since Γ_r is bounded, we can apply the Julia variation to obtain

$$\Delta\text{m.r.}(\Omega_r^*, \Omega_r) = \frac{1}{2\pi}\int_0^{2\pi} \frac{\epsilon d}{|f_r'(e^{i\theta})|} d\theta + o(\epsilon).$$

As $r \to 0, \Omega_r \to \Omega$ and $\Omega_r^* \to \Omega_0^*$ in the sense of kernel convergence. Furthermore, $\dfrac{1}{|f_r'(e^{i\theta})|} \to \dfrac{1}{|f'(e^{i\theta})|}$ as $r \to 0$ except on a countable set, so we have from the dominated convergence theorem that

$$\lim_{r\to 0}\int_0^{2\pi} \frac{1}{|f_r'(e^{i\theta})|} = \int_0^{2\pi} \frac{1}{|f'(e^{i\theta})|} d\theta.$$

It follows then that

$$\Delta\text{m.r.}(\Omega_0^*, \Omega) = \frac{1}{2\pi}\int_0^{2\pi} \frac{\epsilon d}{|f'(e^{i\theta})|} d\theta + o(\epsilon). \tag{11}$$

It follows from results of the first author in [2] that $\Delta\text{m.r.}(\Omega^*, \Omega_0^*) = o(\epsilon)$; hence using (11) we have that (10) also holds in the case that Γ is unbounded.

To show that (7) holds, we will show that for each $\theta \in [0, 2\pi)$ the radial extension $\omega^{**} = (1+\epsilon)f(e^{i\theta})$ of $\omega = f(e^{i\theta})$ is at least as far from the origin as the point ω_*, the point of intersection of $\Gamma^* = \partial\Omega^*$ and the ray $R_t = t\omega : t \geq 1\}$. See Figure 1.

Let $T_j, j = 1$ and 2, denote the left- and right-hand tangent lines to Γ at ω, possibly the same line, say T_1. Let z_j be the projection of the origin onto T_j, l_j the line segment joining the origin to z_j, and d_j the length of l_j. Let θ_j be the acute angle between the ray R_t and the outward normal $n^j(\omega)$ to Γ at ω. Note that the acute angle between R_t and l_j at the origin is also equal to θ_j. Let t_j be the line parallel to T_j which passes through $\omega^{*j} \in \Gamma^*$ and let x_j be the intersection of t_j with R_t. From convexity we have

$$|\omega - \omega_*| \leq \min_j |\omega - x_j|.$$

We note that when there is a corner at ω, i.e., when $x_1 \neq x_2$, or when Γ continues along the tangent line through ω, then $|\omega - \omega_*| = \min_j |\omega - x_j|$, while if Γ does not continue along the tangent line through ω, then $|\omega - \omega_*| < |\omega - x_1|$.

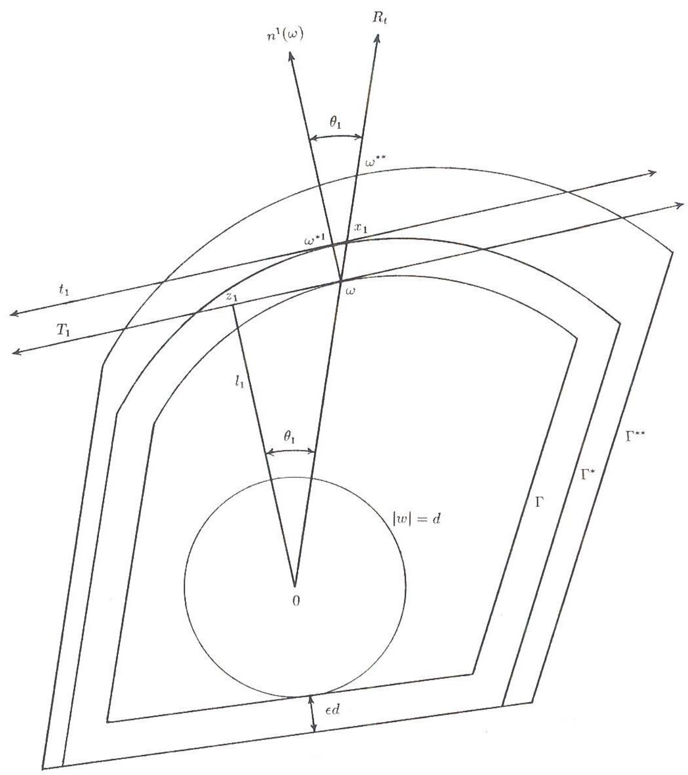

Figure 1:

From the similarity of the triangles $[\omega, \omega^{*j}, x_j]$ and $[0, z_j, \omega]$ we have

$$\frac{\epsilon d}{|\omega - \omega_*|} \geq \max_j \frac{\epsilon d}{|\omega - x_j|} = \max_j \cos\theta_j = \max_j \frac{d_j}{|\omega|} \geq \min_j \frac{d_j}{|\omega|} \geq \frac{d}{|\omega|} \tag{12}$$

which gives

$$\frac{\epsilon d}{|\omega - \omega_*|} \geq \frac{d}{|\omega|}. \tag{13}$$

Thus, $\epsilon|\omega| \geq |\omega - \omega_*|$. Since this is true for all $\omega = f(e^{i\theta}), \theta \in [0, 2\pi)$, we have that (7) holds. Combining (8), (9) and (10) yields

$$\epsilon \geq \frac{\epsilon d}{2\pi} \int_0^{2\pi} \frac{1}{|f'(e^{i\theta})|} d\theta + o(\epsilon). \tag{14}$$

Dividing (14) by ϵd and letting $\epsilon \to 0$ yields the conclusion of the theorem.

We observe that if Γ circumscribes ∂D_d, then for each $\omega \in \Gamma$ we have $d = d_1 = d_2, \theta_1 = \theta_2$ and $|\omega - \omega_*| = |\omega - x_1| = |\omega - x_2|$, so that equality holds in (12) and (13). In fact, if Γ circumscribes ∂D_d, then $\Gamma^* = \Gamma^{**}$ and $\Omega^* = \Omega^{**}$, so that equality holds in (14). ■

Proof of Theorem 2.

Let F_d be the particular convex function, described in the introduction, which circumscribes ∂D_d and whose boundary consists of an arc C_d of ∂D_d symmetric about the positive reals and two lines tangent to ∂D_d at the endpoints of C_d. Applying Theorem 1 in [2], with $\alpha = 1$, to convex functions f which contain the disk D_d gives the subordination

$$\log \frac{f(z)}{z} \prec \log \frac{F_d(z)}{z}, \tag{15}$$

where the subordination in (15) means there exists an analytic function w on D with $|w(z)| \leq |z|$ such that

$$\log \frac{f(z)}{z} = \log \frac{F_d(w(z))}{w(z)}.$$

For $d_r = \min_\theta |f(r(e^{i\theta}))/r|$ this gives

$$\log d_r \geq \log \left| \frac{F_d(r)}{r} \right|. \tag{16}$$

Since, for each $d, \frac{1}{2} \leq d \leq 1, F_d$ is circular symmetric about the positive reals, it takes is minimum modulus on $|z| = r$ at $z = r$, $0 \leq r \leq 1$. Thus, applying Theorem 1 to $f(rz)/r$, using (16) and integrating gives

$$\frac{1}{2\pi} \int_0^{2\pi} \int_0^1 \frac{r\ dr\ d\theta}{|f'(re^{i\theta})|} \leq \int_0^1 \frac{r\ dr}{d_r} \leq \int_0^1 \frac{r^2 dr}{F_d(r)}. \tag{17}$$

From (17) Theorem 2 follows. ■

The functions F_d have been shown to be extremal solutions for growth and subordination problems for the classes K_d, which were studied by Bogucki and Waniurski [4] and by Barnard and Lewis [2], respectively. Also, in a paper related to the authors' earlier work [1] and [3], Waniurski [11] has conjectured that the functions F_d are extremal solutions for an omitted values problem for the classes K_d. Although an explicit closed form for F_d is not known, and appears to be difficult to obtain, the recently developed conformal mapping package CONFPACK [7] can be used to numerically approximate F_d and, subsequently, the integral in the right-hand side of (17). We include a table of values.

Table 1

d	β	Integral
0.500	-0.50000	$0.8333\bar{3}$
0.525	-0.47473	0.80669
0.550	-0.44895	0.78113
0.575	-0.42268	0.75692
0.600	-0.39589	0.73411
0.625	-0.36855	0.71263
0.650	-0.34059	0.69241
0.675	-0.31192	0.67336
0.700	-0.28243	0.65540
0.725	-0.25200	0.63845
0.750	-0.22046	0.62242
0.775	-0.18757	0.60724
0.800	-0.15308	0.59286
0.825	-0.11658	0.57921
0.850	-0.07757	0.56796
0.875		
0.900		
0.925	0.06499	
0.950	0.12922	0.52013
0.975	0.21574	0.50985
1.000	0.50000	0.50000

For $\partial F_d(D)$, let ω_d denote the endpoint of C_d in the upper half-plane. The variable $\beta, -\frac{1}{2} \leq \beta \leq \frac{1}{2}$, in Table 1 parameterizes the family of functions $F_d, d = d(\beta)$. If β is positive, then the domain $F_d(D)$ is bounded; otherwise $F_d(D)$ is unbounded. The argument of ω_d is $\pi/2 + \beta\pi$. Also, if β is positive, then $2\beta\pi$ is the interior angle for $F_d(D)$ at the finite intersection point of the lines which bound $F_d(D)$, tangent to ∂D_d at the endpoints of C_d; otherwise $2\beta\pi$ is the interior angle for $F_d(D)$ at the point of infinity.

The conformal mapping package CONFPACK restricts its applications to bounded domains. For $\beta > 0$, the domains $F_{d(\beta)}(D)$ were computed directly. For $\beta < 0$ the domains $F_{d(\beta)}(D)$ were computed by applying a bilinear transformation to $F_{d(-\beta)}$. At $\beta = 0$ the domain $F_{d(0)}(D), d(0) \approx 0.8941$, is unbounded and its boundary is composed of an arc $C_{d(0)}$ of the circle $\partial D_{d(0)}$ and two lines, parallel to the negative real axis and tangent to the circle $\partial D_{d(0)}$ at the points of $C_{d(0)}$, $\pm id(0)$. The gap in Table 1 occurs because for β near 0 the elongation of $F_{d(\beta)}(D)$ causes crowding on ∂D of the pre-images of endpoints of the boundary segments defining $F_{d(\beta)}(D)$, which causes CONFPACK to fail to converge.

Proof of Theorem 3

It is well known [9] that $f \in S^*(\alpha)$ if and only if

$$\frac{zf'(z)}{f(z)} \prec \frac{1+(1-2\alpha)z}{1-z}. \tag{18}$$

Thus, applying Littlewood's subordination theorem [6], we have for $z = e^{i\theta}$

$$\begin{aligned}
\frac{1}{2\pi}\int_0^{2\pi} \frac{1}{|f'(z)|} d\theta &= \frac{1}{2\pi}\int_{|z|=1} \left| \frac{f(z)}{zf'(z)} \cdot \frac{z}{f(z)} \right| |dz| \le \frac{1}{d}\frac{1}{2\pi}\int_0^{2\pi} \left| \frac{f(z)}{zf'(z)} \right| d\theta \\
&\le \frac{1}{d}\frac{1}{2\pi}\int_0^{2\pi} \left| \frac{1-z}{1+(1-2\alpha)z} \right| d\theta = \frac{1}{d}\frac{4}{\pi}\frac{\arctan\sqrt{2\alpha-1}}{\sqrt{2\alpha-1}} = \frac{A(\alpha)}{d}.
\end{aligned}$$

Sharpness follows by using $f_n(z) = z(1-z^n)^{\frac{2(\alpha-1)}{n}}$ and observing that for $d_n = \min\limits_{\theta}|f_n(e^{i\theta})| = 2^{\frac{2(\alpha-1)}{n}}$ we have

$$\lim_{n\to\infty} \frac{d_n}{2\pi}\int_0^{2\pi} \left| \frac{1}{f_n'(e^{i\theta})} \right| d\theta = A(\alpha)$$

as claimed. We note $A(\frac{1}{2}) = 4/\pi$. ■

4 Note to Theorem 1

Alternately stated, Theorem 1 gives an upper bound for the integral mean $I_{-1}(f')$ for $f \in K$ in terms of the minimum distance to the envelope of tangent lines to the boundary of $f(D)$. An analogous theorem can be proved which gives a lower bound for the integral mean $I_{-1}(f')$ for $f \in K$ in terms of the maximum distance to the envelope of tangent lines to the boundary of $f(D)$. More precisely, in the notation of the proof of Theorem 1, for each $\omega \in \Gamma$, let $T_j, j = 1$ and 2, denote the left- and right-hand tangent lines to Γ at ω, possibly the same line T_1. Let z_j be the projection of the origin on T_j, l_j the line segment joining the origin to z_j, and d_j the length of l_j. Let $d^* = d^*(f) = \sup\limits_{\omega\in\Gamma} d_j$.

We have

Theorem 1$*$ *Let $f \in K$. Then*

$$\frac{1}{d^*} \leq \frac{1}{2\pi} \int\limits_0^{2\pi} \left| \frac{1}{f'(e^{i\theta})} \right| d\theta \tag{19}$$

with equality holding in (19) *if the boundary of $f(D)$ circumscribes ∂D_{d^*}.*

Proof.

The major step in the proof is to show, using the convexity of the domain $\Omega = f(D)$, there exist two varied domains Ω^{**} and Ω^{***} which satisfy the inequality

$$\Omega^{**} \subseteq \Omega^{***}. \tag{20}$$

The domain Ω^{**} is constructed exactly the same as in Theorem 1, i.e., it is the $(1+\epsilon)$ radial expansion of Ω. The domain Ω^{***} is constructed as an outward normal expansion of Ω, just as Ω^* was, only we take the constant normal distance to be ϵd^* instead of ϵd. An analogous argument can be given to show that an inequality similar to (12) holds, only the inequality senses are all reversed. In this case, the first inequality in the analog of (12) is more delicate, but it can be verified by considering a sequence of convex polygonal domains which converge to Ω and verifying the inequality for the polygonal domains. Then, (20) holds and a comparison of the mapping radii of Ω^{**} and Ω^{***} yields

$$\epsilon \leq \frac{\epsilon d^*}{2\pi} \int\limits_0^{2\pi} \frac{1}{|f'(e^{i\theta})|} d\theta + o(\epsilon)$$

from which Theorem 1$*$ follows. ■

If we let $d_* = d_*(f) = \min\limits_{\omega\in\Gamma} d_j$, then combining both (3) and (19) we have

Corollary. *Let $f \in K$. Then,*

$$\frac{1}{d^*} \leq \frac{1}{2\pi} \int\limits_0^{2\pi} \left| \frac{1}{f'(e^{i\theta})} \right| d\theta \leq \frac{1}{d_*}. \tag{21}$$

Equality holds across (21) *if for some* $d, \frac{1}{2} \leq d \leq 1, \partial f(D)$ *circumscribes* ∂D_d.

Remark. For $f \in K$ let $L =$ length $\partial f(D)$ and let F denote the inverse of f. The isoperimetric inequaltiy states

$$\frac{2\pi d_*}{L} \leq 1.$$

Schwarz's inequality, applied to the integral means in (21) – rewritten in terms of F, yields

$$\frac{2\pi}{L} \leq \frac{1}{2\pi} \int_0^{2\pi} \left| \frac{1}{f'(e^{i\theta})} \right| d\theta. \tag{22}$$

Thus, (21) combined with (22) gives an intermediate term to the isoperimetric inequality.

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