Polynomials with non-negative coefficients

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1 Introduction

Can a conjugate pair of zeros be factored from a polynomial with nonnegative coefficients so that the resulting polynomial still has nonnegative coefficients?

This question has attracted considerable attention during the last few years because of its seemingly elementary nature and its potential for applications in number theory and control theory. A proposed answer to this question arose as a conjecture out of the work in [6], where explicit bounds were determined for the constants occurring in the Beauzamy-Enflo generalization [1], [2] of Jensen's Inequality. An affirmative answer to this question was conjectured, independently, by B. Conrey in connection with some of his work in number theory. Conrey announced the conjecture at the annual West Coast Number Theory Conference held in December 1987.

The main theorem in this note gives a positive answer to the question. This will be proved in section 2. The principle ingredients of the proof are an idea from index theory, classical properties of polynomials and a significant lemma (Lemma 2.1) which we prove in section 3. This lemma states some strong consequences for the case of equality holding in Descartes' Rule of Signs (see [5]). We also prove a corollary of the main theorem which describes the region into which certain zeros can be moved while preserving the nonnegativity of the coefficients.

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Consider the polynomial defined by $P_N(z) = 1 + z^N$. We note that

$$P_N(z) = \prod_{l=0}^{N-1} \left(1 - \frac{z}{e^{i\left(\frac{\pi}{N}+\frac{2\pi l}{N}\right)}}\right) = (1 - 2\cos\theta_0 z + z^2) \sum_{k=0}^{N-2} b_k z^k \tag{1}$$

has $b_k > 0, 0 \leq k \leq N-2$, if $\theta_0 = \pi/N$, while any other choice for θ_0 produces a factor with some negative b_k coefficients.

Thus, one initial suggestion for the general question of factoring out a conjugate pair of zeros was to factor out a pair of zeros with greatest real part. The authors have used fairly straightforward arguments to show that if the degree of the polynomial is less than or equal to 5, then a conjugate pair of zeros of greatest real part can be factored out and the resulting polynomial will still have nonnegative coefficients. However, if

$$p(z) = 140 + 20z + z^2 + 1000z^3 + 950z^4 + 5z^5 + 20z^6$$

with $z_0, \overline{z}_0$ approximately equal to $0.392 + 6.390i, 0.392 - 6.392i$, resp., as the conjugate pair of zeros of greatest real part, then the resulting factors are $(z - z_0)(z - \overline{z}_0)$ and $(140.1 + 22.3z - 1.53z^2 + 1000z^3 + 966.7z^4)$.

The other choice, for which zeros to factor out, suggested by the example P_N in (1), is the pair determined by the zero in the upper half-plane with smallest positive argument. Indeed, in this paper we prove the following:

Theorem 1.1 *Let p be a polynomial of degree N, $p(0) = 1$, with nonnegative coefficients and with zeros $z_1, z_2, \ldots, z_N$. For $t \geq 0$ write*

$$p_t(z) = \prod_{\substack{1\leq j\leq N \\ |Arg z_j| > t}} \left(1 - \frac{z}{z_j}\right)$$

Then, if $p_t \neq p$, all of the coefficients of p_t are positive.

Remark 1 *We note that a slightly sharper version of the Theorem 1.1 is true. Suppose that the given polynomial p has nonnegative coefficients and that we divide it by a single quadratic factor corresponding to any conjugate pair of zeros of smallest argument in magnitude. Then the quotient polynomial has nonnegative coefficients. Furthermore, if the conjugate pair of zeros is unique and simple or if p has strictly positive coefficients then* *the quotient polynomial will have strictly positive coefficients. The example $p(z) = (1 + z^2)^2$ shows the necessity of these conditions. The sharpened version of the theorem under the uniqueness hypothesis follows from the main theorem. Now let us consider the case when several complex conjugate pairs of zeros of p, say $\{z_i, \bar{z}_i\}_{i=1,\ldots,k}$, have the smallest argument in magnitude and assume that all coefficients of p are strictly positive. A local perturbation can be made of z_1 and $\bar{z}_1$ to w_1 and $\bar{w}_1$ such that the argument of w_1 is less than that of z_1. Under the local perturbation the polynomial p changes to q which also has strictly positive coefficients. By Theorem 1.1, $q/(z - w_1)(z - \bar{w}_1)$ has strictly positive coefficients. Since $p/(z - z_1)(z - \bar{z}_1) = q/(z - w_1)(z - \bar{w}_1)$ the desired conclusion follows.*

Corollary 1.1 *Let*

$$p(z) = \sum_{k=0}^{N} a_k(z_0) z^k = (z - z_0)(z - \bar{z}_0) \sum_{k=0}^{N-2} b_k z^k$$

be a real polynomial with $a_k(z_0) \geq 0$. Let z_0 be a zero of p in the upper half plane with smallest argument. If z_1 is any complex number such that

$$|z_1| \geq |z_0| \text{ and } Re\{z_1\} \leq Re\{z_0\}$$

then

$$\sum_{k=0}^{N} a_k(z_1) z^k = (z - z_1)(z - \bar{z}_1) \sum_{k=0}^{N-2} b_k z^k$$

has $a_k(z_1) \geq a_k(z_0)$.

Proof: The proof follows directly by noting, for $z_0 = re^{i\theta} = x + iy$,

$$\begin{aligned} \sum_{k=0}^{N} a_k(z_0) z^k &= (r^2 - 2rz\cos\theta + z^2) \sum_{k=0}^{N} b_k z^k \\ &= (x^2 + y^2 - 2xz + z^2) \sum_{k=0}^{N} b_k z^k \end{aligned}$$

and comparing coefficients.

It has been recently shown by R. Evans and P. Montgomery[4] that in the special case when p is the polynomial P_N defined in (1) that the corresponding reduced p_t polynomials are all strictly unimodal (in their coefficients). Also, R. Evans and J. Greene [3] have shown for polynomials p of the form $p(z) = (z^{Mk}-1)/(z^k-1)$ that the corresponding reduced p_t polynomials all have positive coefficients.

2 Proof of theorem 1.1

We note to prove Theorem 1.1 it suffices, by a successive reduction argument, to show that if we remove from p a conjugate pair of zeros of smallest argument in the absolute value, then the resulting polynomial still has non-negative coefficients.

Proof.

Let

$$p(z) = \sum_{k=0}^{N} a_k z^k = (1 - \frac{2\cos\theta_0 z}{r_0} + \frac{z^2}{r_0^2}) \sum_{k=0}^{N-2} b_k z^k \tag{2}$$

with $a_k \geq 0, 0 \leq k \leq N$. We may assume $a_N > 0$. Let $\{r_\ell e^{i\theta_\ell}\}$ be the zero set of p with $0 < \theta_0 \leq |\theta_\ell|$ for each ℓ. We may assume $r_0 = 1$ in (2) by using $p(r_0 z)$. Since the coefficients of p depend continuously on the zeros of p, we may assume that p has only one zero, say $z_0 = e^{i\theta_0}$, on the ray $\{re^{i\theta_0} : 0 < r < \infty\}$ and that this zero is simple. Otherwise, a sequence of polynomials with this property can be chosen to converge to the desired polynomial with the appropriate inequalities holding. We will prove that $b_k > 0$ for $0 \leq k \leq N-2$.

Now

$$\begin{aligned}\sum_{k=0}^{N-2} b_k z^k &= \frac{p(z)}{1 - 2\cos\theta_0 z + z^2} \\ &= \sum_{k=0}^{\infty} \frac{\sin(k+1)\theta_0}{\sin\theta_0} z^k \sum_{k=0}^{N} a_k z^k = \sum_{k=0}^{N-2} \left(\sum_{\ell=0}^{k} \frac{\sin(\ell+1)\theta_0}{\sin\theta_0} a_{k-\ell} \right) z^k.\end{aligned}$$

This gives

$$b_k = \sum_{\ell=0}^{k} \frac{\sin(\ell+1)\theta_0}{\sin\theta_0} a_{k-\ell}. \tag{3}$$

Noting that

$$z^N p(1/z) = \sum_{k=0}^{N} a_{N-k} z^k = \left(z^2 - 2\cos\theta_0 z + 1\right) \sum_{k=0}^{N-2} b_{N-2-k} z^k$$

it will suffice to show that $b_k > 0$ for $0 \le k \le [\frac{N-2}{2}]$. From (3) if $\sin(\ell+1)\theta_0 > 0$ for $\ell = 0, 1, \cdots, k$, i.e., when $0 \le \theta_0 < \pi/(k+1)$ it

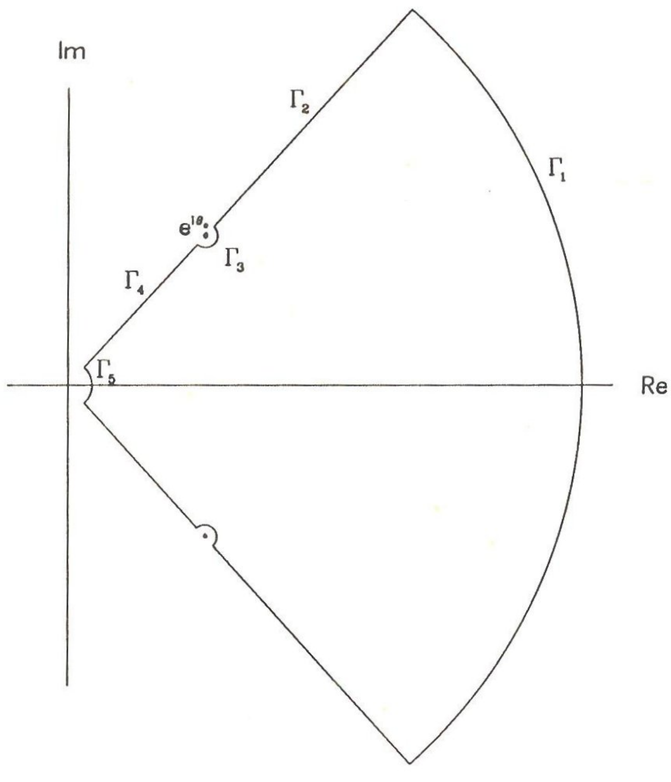

Figure 1:

follows that $b_k > 0$. Also, if all the zeros of p are in the closure of the left half plane then the result is clear since p can be put into the form: $p(z) = \prod(1 - \frac{2\cos\theta_\ell}{r_\ell} z + \frac{z^2}{r_\ell^2}) \prod(1 + \frac{z}{r_m})$. Thus, we may assume

$$\pi/\left(\left[\frac{N-2}{2}\right] + 1\right) = \pi/\left(\left[\frac{N}{2}\right]\right) \le \theta_0 < \pi/2. \tag{4}$$

We will assume that there is a k with

$$b_k \leq 0 \tag{5}$$

to reach a contradiction. Consider the function defined by

$$F(z) = z^{-k-1}p(z) = \sum_{\ell=0}^{N} a_\ell z^{\ell-k-1}$$

having the same zeros as p.

We observe from the hypothesis that F has no zeros between the rays defined by $\{te^{i\theta_0} : 0 < t < \infty\}$ and $\{te^{-i\theta_0} : 0 < t < \infty\}$. Motivated by this, let $\Gamma = \Gamma^+ \bigcup \overline{\Gamma}^+$ be the curve shown in Figure 1, which is symmetric about the reals with $\Gamma^+ = \bigcup_{\ell=1}^{5} \Gamma_\ell$ where $\Gamma_1 = \{Re^{i\theta} : 0 \leq \theta \leq \theta_0\}$, $\Gamma_2 = \{te^{i\theta_0} : 1+s \leq t \leq R\}$, $\Gamma_3 = \{z_0+se^{i\theta} : \theta_0-\pi \leq \theta \leq \theta_0\}$, $\Gamma_4 = \{te^{i\theta_0} : r \leq t \leq 1-s\}$ and $\Gamma_5 = \{re^{i\theta} : 0 \leq \theta \leq \theta_0\}$ and r, s, and $1/R$ will be chosen sufficiently small. We will show that under the assumption (5) the index of F with respect to Γ about any interior point is positive, implying the existence of a zero of F inside Γ. This would contradict that θ_0 is the smallest positive argument of the zeros of p in the upper half plane.

The function F has the form

$$F(z) = a_0 z^{-k-1} + \cdots + a_{k+1} + a_{k+2}z + \cdots a_N z^{N-k-1}. \tag{6}$$

From symmetry we can examine $\Delta \arg F(z)$, the change in the argument of F, as z traverses Γ^+. Let Δ_ℓ equal $\Delta \arg F(z)$ as z traverses Γ_ℓ. For R sufficiently large the term $a_N z^{N-k-1}$ in (6) dominates so that $\Delta_1 = (N-k-1)\theta_0 + O(\frac{1}{R})$. For r sufficiently small and positive the term $a_0 z^{-k-1}$ in (6) dominates so that $\Delta_5 = -(-k-1)\theta_0 + O(r) = (k+1)\theta_0 + O(r)$, noting the clockwise transversal of Γ_5. Hence,

$$\Delta_1 + \Delta_5 = N\theta_0 + O\left(\frac{1}{R}\right) + O(r) \tag{7}$$

for $1/R$ and r sufficiently small. Let $\Gamma_6 = \{te^{i\theta_0} : r \leq t \leq R\}$, noting that $\Gamma_2 \cup \Gamma_3 \cup \Gamma_4$ approaches Γ_6 as $s \to O$. For notational convenience let $\hat{\Delta}_6$ denote $\Delta_2 + \Delta_4 + \lim \Delta_3$ as $s \to 0$. We observe that as t crosses 1 the change in argument of $-\pi$ accounts for the change in argument determined by traversing Γ_3 for s sufficiently small, i.e., $\Delta_3 = -\pi + O(s)$.

Since a bound for $\hat{\Delta}_6$ can be determined by the maximum number of times the image of Γ_6 crosses the real axis we consider $\alpha(f_0)$, the number of real positive zeros of f_0 defined by

$$
\begin{aligned}
f_0(t) &= Im\left[t^{k+1}F(te^{i\theta_0})\right] && (8)\\
&= -\sum_{\ell=0}^{k} a_\ell\left[\sin(k+1-\ell)\theta_0\right]t^\ell + \sum_{\ell=k+2}^{N} a_\ell\left[\sin(\ell-k-1)\theta_0\right]t^\ell.
\end{aligned}
$$

Descartes' Rule of Signs says that $\alpha(f_0)$ is bounded above by $\gamma(f_0)$, the number of sign changes in the coefficients of f_0. Since, $a_\ell \geq 0$ for each $\ell, \gamma(f_0)$ is determined by the number of sign changes in $\sin \ell\theta_0$ as ℓ goes from $-k-1$ to $N-k-1$, i.e., over a range of N. It is clear that there exists an m such that

$$\frac{m\pi}{N} < \theta_0 \leq \frac{(m+1)\pi}{N}. \tag{9}$$

Since the number of sign changes as $\ell\theta_0$ varies over an $N\theta_0$ range is determined by $N\theta_0/\pi$, it follows that $\gamma(f_0) \leq m+1$. In counting $\alpha(f_0)$ and $\gamma(f_0)$ if $a_0 \sin(k+1)\theta_0 = 0$, then we factor out the leading power of t^ℓ in f_0. So, we are only considering positive zeros of f_0.

In the case when $\alpha(f_0)$ is strictly less than m, say m_1 with $0 \leq m_1 < m$, it follows that $\hat{\Delta}_6 < m_1\pi + \pi \leq m\pi$. Thus, by using (7), (8) and (9) we obtain

$$\Delta \arg F(z) \geq 2\sum_{\ell=1}^{5} > 2(N\theta_0 - (m_1+1)\pi) \geq 0.$$

This would imply that Γ encloses a zero of F, contradicting the construction of Γ.

Now we consider the remaining cases, i.e., when $\gamma(f_0) = m$ or $m+1$ and $\alpha(f_0) \geq m$. If $\gamma(f_0) - \alpha(f_0) = 1$, then one of the positive zeros of f_0 occurs with even multiplicity. This can be easily seen by observing that the sign of $f_0(t)$ for large t can be determined by multiplying the sign of the constant term of f_0 by $(-1)^{\gamma(f_0)}$ and, alternatively, by multiplying the sign of the constant term of f_0 by $(-1)^{\alpha(f_0)}$ if all of the positive zeros of f_0 have odd multiplicity. But, a positive zero of f_0 with even multiplicity does not correspond to a proper crossing of the real axis by $f(te^{i\theta_0})$ and, hence, $\hat{\Delta}_6 \leq m\pi$ in this case, which is impossible.

The only remaining case is when $\alpha(f_0) = \gamma(f_0) = m$ or $m+1$. We shall need an involved technical lemma, which we state and prove asLemmas 3.3

and 3.4 in the next section. To complete the proof we state Lemma 3.3 as Lemma 2.1 in the following form:

Lemma 2.1 *Let g and h be two real polynomials with $h(z) = a_\ell z^\ell + \cdots + a_n z^n$ where $a_\ell > 0$ and the degree of g is less than ℓ with the last coefficient of g being negative. If p is the polynomial defined by $p = g + h$ and the number of sign charges in the coefficients of p equals the number of real positive zeros of p then $h(z_j)$ is positive at each of the zeros, z_j, of p.*

We observe that the function f_0 defined in (8) satisfies the hypothesis of Lemma 2.1 with $f_0 = g + h$ where $g(t) = -\sum_{l=1}^{k} a_\ell [\sin (k+1-\ell)\theta_0]\, t^\ell$ and $h(t) = \sum_{l=k+2}^{N} a_\ell [\sin (l-k-1)\theta_0]\, t^\ell$. Since $\alpha(f_0) = \gamma(f_0) = m$ or $m+1$, it follows, from a partitioning argument, that the last nonzero term of $g(t)$ is negative and that the first term of $h(t)$ is positive, for otherwise we would have $\gamma(f_0) \leq m-1$, which would reduce to the previous case. Now, since $f_0(1) = 0$, Lemma 2.1 gives that $h(1) > 0$, which implies that $g(1) < 0$. But, using (3) it follows that $g(1) = -b_k \sin \theta_0$, which contradicts the assumption in (5). Our main theorem follows.

3 Lemmas on Zeros, Critical Points, and Sign Changes

Let $f(x)$ denote a polynomial with real coefficients. Write

$$f(x) = \pm\,(p_0(x) - p_1(x) + p_2(x) - p_3(x) + \cdots)$$

where

$$\begin{aligned} p_i(x) &= c_{i0}x^{n_{i-1}} + c_{i1}x^{n_{i-1}+1} + \cdots + c_{i,m_i}x^{n_{i-1}+m_i} \\ n_{-1} &= 0,\ n_i = n_{i-1} + m_i + 1, \\ c_{ij} &\geq 0\ , i \geq 0, j \geq 0. \end{aligned}$$

Notation *$\alpha(f)$ denotes the number of strictly positive real zeros, $\gamma(f)$ denotes the number of sign changes, $\tilde{\beta}(f)$ denotes the number of strictly positive critical points,*

$$\beta(f) = \begin{cases} 0 & \text{if} \qquad f \equiv c_{00} \\ \tilde{\beta}(f) + 1 & \text{if} \qquad p_0(x) \equiv c_{00} \\ \tilde{\beta}(f) & \text{otherwise.} \end{cases}$$

Let $0 < x_1(f) < x_2(f) < x_3(f) < \cdots$ denote the distinct positive real zeros of f. Let $y_i(f)$ denote the critical point between $x_i(f)$ and $x_{i+1}(f)$ (in the case where $\alpha(f) = \gamma(f)$).

Lemma 3.1 *(i) $\alpha(f) \leq \beta(f) \leq \gamma(f)$.*

(ii) If $\alpha(f) = \gamma(f)$, then $\alpha(f^{(k)}) = \gamma(f^{(k)})$ for all k such that $f^{(k)} \not\equiv 0$.

Proof.

(i) This is an immediate consequence of Rolle's Theorem and Descartes' Rule of Signs.

(ii) It needs only to be shown that $\alpha(f^{'}) = \gamma(f^{'})$

Case 1: $p_0(x) \neq c_{00}$. Then $\alpha(f') = \tilde{\beta}(f) = \beta(f) = \gamma(f) = \gamma(f^{'})$.
Case 2: $p_0(x) = c_{00}$. Then, $\alpha(f^{'}) = \tilde{\beta}(f) = \beta(f) - 1 = \gamma(f) - 1 = \gamma(f^{'})$

□

Remark 2 *The remaining lemmas will each contain the hypothesis that $\alpha(f) = \gamma(f)$. It follows from Lemma 3.1(i) that $\alpha(f) = \beta(f)$. Therefore, the positive real zeros and critical points of f must strictly interlace. We emphasize again that all coefficients of each of the polynomials $p_i(x)$ appearing below are nonnegative. It will be useful to the reader at this stage to note that the essentially different cases are,*

(i) $p_0 \not\equiv c_{00} > 0$

(ii) $p_0 \equiv c_{00} > 0$

(iii) $c_{00} = 0$

It also may be beneficial to draw rough sketches for each of the cases.

Lemma 3.2 *Let $f(x) = p_0(x) - p_1(x) + h(x)$, where $h(x) = p_2(x) - p_3(x) + \cdots$. Suppose that $\alpha(f) = \gamma(f)$, then $h(x_i(f)) > 0$ for all i.*

Proof. Let $p_1(x) = c_n x^n + c_{n+1}x^{n+1} + \cdots + c_r x^r$, where $c_r > 0$ and $n = 0$ is allowed. (If $n = 0$, then $p_0(x) \equiv 0$.)

Since the lemma is trivial if $\alpha(h) = 0$, let us assume that $\alpha(h) > 0$. Observe that $x_1(h^{(j)})$ is defined for $0 \leq j \leq r+1$, since $h^{(j)}(0) = 0$ for $0 \leq j \leq r$. Now $f^{(r)}(x) = -r!c_r + h^{(r)}$ has a critical point at $x_1(h^{(r+1)})$. Since $\alpha(f^{(r)}) = \gamma(f^{(r)})$ (Lemma 3.1(ii)), it follows that $x_1(f^{(r)}) < x_1(h^{(r+1)})) < x_1(h^{(r)})$, and $f^{(r)}(x_1(h^{(r+1)})) > 0$. Furthermore, since $f^{(r)} < h^{(r)}$ it follows that $x_1(f^{(r)}) < x_1(h^{(r+1)}) < x_2(f^{(r)}) < x_1(h^{(r)})$. Now $f^{(r-1)}$ has critical points at $x_1(f^{(r)})$ and $x_2(f^{(r)})$ and since $\alpha(f^{(r-1)}) = \gamma(f^{(r-1)})$ it follows that $x_1(f^{(r)}) < x_1(f^{(r-1)}) < x_2(f^{(r)})$. Therefore, $f^{(r-1)}(x_2(f^{(r)})) > 0$, and since $f^{(r-1)} < h^{(r-1)}$, it now follows that $x_2(f^{(r)}) < x_2(f^{(r-1)}) < x_1(h^{(r-1)})$. Thus, $x_1(f^{(r-1)}) < x_2(f^{(r)}) < x_2(f^{(r-1)}) < x_1(h^{(r-1)})$. An easy induction then shows that

$$x_1(f^{(j)}) < x_2(f^{(j)}) < x_1(h^{(j)}), n \leq j \leq r-1.$$

Now let $p_0(x) = d_0 + d_1 x + \cdots + d_{n-1}x^{n-1}$. Since it suffices to prove the lemma in the generic case, we assume that $d_i > 0;\ 0 \leq i \leq n-1$. Since $f^{(n-1)}$ has critical points at $x_1(f^{(n)})$ and $x_2(f^{(n)})$ and $\alpha(f^{(n-1)}) = \gamma(f^{(n-1)})$, it follows that $f^{(n-1)}(x_1(f^{(n)})) < 0$ and $f^{(n-1)}(x_2(f^{(n)})) > 0$. Since $f^{(n)}(x) < h^{(n)}(x)$ for $x > 0$, it now follows that for all $x \geq x_1(f^{(n)})$,

$$\begin{aligned} f^{(n-1)}(x) &= f^{(n-1)}\left(x_1\left(f^{(n)}\right)\right) + \int_{x_1\left(f^{(n)}\right)}^{x} f^{(n)}(t)dt \\ &< \int_{x_1\left(f^{(n)}\right)}^{x} h^{(n)}(t)dt < h^{(n-1)}(x). \end{aligned}$$

By a previous argument, we have that $x_1(f^{(n)}) < x_2(f^{(n)}) < x_1(h^{(n)}) < x_1(h^{(n-1)})$. Then, since $f^{(n-1)}(x) < h^{(n-1)}(x)$ for $x \geq x_1(f^{(n)})$ and since $f^{(n-1)}(x_2(f^{(n)})) > 0$, we have that $x_3(f^{(n-1)}) < x_1(h^{(n-1)})$. Therefore,

$$\begin{aligned} x_1\left(f^{(n-1)}\right) < x_1\left(f^{(n)}\right) < x_2\left(f^{(n-1)}\right) < x_2\left(f^{(n)}\right) \\ < x_3\left(f^{(n-1)}\right) < x_1\left(h^{(n-1)}\right). \end{aligned}$$

Now suppose that for some j with $0 < j \leq n-1$,

$$x_1(f^{(j)}) < x_2(f^{(j)}) < x_3(f^{(j)}) < x_1(h^{(j)})$$

and $f^{(j)}(x) < h^{(j)}(x)$ for all $x > y_1(f^{(j)})$.

Then, exactly the same reasoning as above applies and it follows that

$$x_1\left(f^{(j-1)}\right) < x_2\left(f^{(j-1)}\right) < x_3\left(f^{(j-1)}\right) < x_1\left(h^{(j-1)}\right) \qquad (*^j)$$

and $f^{(j-1)}(x) < h^{(j-1)}(x)$ for all $x > y_1(f^{(j-1)})$. Therefore, $(*^j)$ is true for $0 \leq j \leq n-1$.

In particular, $(*^0)$ shows that $x_1(f) < x_1(h)$. Thus, $-p_0(x_1(f))+p_1(x_1(f)) = h(x_1(f)) > 0$. By Lemma 3.1 $p_1 - p_0$ has only one zero. Therefore, $(p_1-p_0)(x)$ is monotone increasing for all $x > x_1(p_1 - p_0)$. Since $(p_1 - p_0)(x_1(f)) > 0$, we have $x_1(f) > x_1(p_1 - p_0)$ and it follows that $h(x_j(f)) > 0$ for all j. $\square$

Lemma 3.3 *Let $f = (p_0 - p_1) + (p_2 - p_3) + \cdots + (p_{2k+2} - p_{2k+3}) + (p_{2k+4} - p_{2k+5}) + (p_{2k+6} - \cdots)$, where $g_0 = p_0 - p_1,\ g_1 = p_2 - p_3, \cdots, g_{k+1} = p_{2k+2} - p_{2k+3},\ h = p_{2k+4} - p_{2k+5} + p_{2k+6} - \cdots$. Suppose that $\alpha(f) = \gamma(f)$, then $h(x_i(f)) > 0$ for all i.*

Proof. We may assume that $\gamma(h) \neq 0$, for otherwise the result is trivial. Hence, $\gamma(h^{(j)}) \neq 0$ for $0 \leq j \leq \deg(p_{2k+3})$. Let $n_j = \deg(g_j)$ and $m_j = n_j + 1$ for $0 \leq j \leq k+1$. Let $m_{-1} = 0$.

We now regard k as fixed and proceed by induction on ℓ, where $-1 \leq \ell \leq k$. From the previous lemma it follows that

$$x_1(f^{(m_k)}) < x_2(f^{(m_k)}) < x_3(f^{(m_k)}) < x_1(h^{(m_k)})$$

and $(h^{(m_k)} - f^{(m_k)})(x)$ is positive and monotone increasing for $x \geq y_1(f^{(m_k)})$

We now make the following induction hypothesis: For some ℓ with $0 \leq \ell \leq k$,

$$x_1(f^{(m_\ell)}) < \cdots < x_{2(k-\ell)+3}(f^{(m_\ell)}) < x_1(h^{(m_\ell)})$$

and $(h^{(m_\ell)} - f^{(m_\ell)})(x)$ is positive and monotone increasing for $x \geq y_{2(k-\ell)+1}(f^{(m_\ell)})$.

Let $p_{2\ell+1}(x) = a_0 x^s + a_1 x^{s+1} + \cdots + a_r x^r$, where $r = n_\ell = m_\ell - 1$. Now $f^{(r)}(x) = \int_0^x f^{(m_\ell)}(t)dt - c$, where c is a nonnegative constant. Observe that $f^{(r)}(x)$ is positive for

$$x_{2(k-\ell)+3}(f^{(r)}) < x < x_{2(k-\ell)+4}(f^{(r)}). \qquad (10)$$

Therefore, for x satisfying (10)

$$\begin{aligned} 0 \le f^{(r)}(x) &= \int_{x_{2(k-\ell)+3}}^{x} f^{(m_\ell)}(t)dt \\ &\le \int_{x_{2(k-\ell)+3}}^{x} h^{(m_\ell)}(t)dt \end{aligned}$$

(by the induction hypothesis since $x_{2(k-\ell)+3}(f^{(r)}) > y_{2(k-\ell)+1}(f^{(m_\ell)})$)

$$< h^{(r)}(x_{2(k-\ell)+3}(f^{(r)})) + \int_{x_{2(k-\ell)+3}}^{x} h^{(m_\ell)}(t)dt.$$

Therefore, $x_{2(k-\ell)+4}(f^{(r)}) < x_1(h^{(r)})$ and since $\alpha(f^{(r)}) = \gamma(f^{(r)})$, it follows that

$$\begin{array}{llll} x_1(f^{(r)}) & < x_1(f^{(m_\ell)}) & < x_2(f^{(r)}) & < \cdots \\ & < x_{2(k-\ell)+3}(f^{(m_\ell)}) & < x_{2(k-\ell)+4}(f^{(r)}) & < x_1(h^{(r)}) \end{array} \tag{11}$$

and $(h^{(r)} - f^{(r)})(x)$ is positive and monotone increasing for $x > y_{2(k-\ell)+2}(f^{(r)})$.

A repetition of the above argument shows that (11) remains true when r is replaced by j and m_ℓ by $(j+1)$ for $s \le j \le r$ and also shows that $x_{2(k-\ell)+5}(f^{(m_{\ell-1})}) < x_1(h^{(m_\ell - 1)})$. This verifies the induction hypothesis for $(\ell - 1)$.

In particular, when $\ell = -1$ we have $x_1(f) < x_2(f) < \cdots < x_{2k+5}(f) < x_1(h)$ and $(h-f)(x) > 0$ for $x \ge y_{2k+3}(f)$. Since $h(x) > 0$ for $0 < x < x_1(h)$ and since $\sum_{j=0}^{k+1} g_j(x) < 0$ for $x \ge y_{2k+3}(f)$ and $y_{2k+3}(f) < x_1(h)$, we finally have $h(x_i(f)) > 0$ for all i. $\square$

The same argument establishes

Lemma 3.4 *Let* $f = -p_0 + (p_1 - p_2) + \cdots + (p_{2k-1} - p_{2k}) + (p_{2k+1} - p_{2k+2}) + \cdots$ *Let* $h = p_{2k+1} - p_{2k+2} + \cdots$. *Suppose that* $\alpha(f) = \gamma(f)$, *then* $h(x_i(f)) > 0$ *for all* i.

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References

[1] Beauzamy, B. *Jensen's inequality for polynomials with concentration at low degrees.* Numer. Math. No. 49 (1986) pp. 221-225.

[2] Beauzamy, B. and Enflo, P. *Estimations de produits de polynômes.* J. Number Theory. No. 21 (1985) pp. 390-412.

[3] Evans, R. and Greene, J. *Polynomials with positive coefficients whose zeros have modulus one.* preprint.

[4] Evans, R. and Montgomery, P. *Elementary Problem Proposal for American Mathematical Monthly.* Amer. Math. Mon. to appear.

[5] Polya, G. and Szego, G. *Problems and theorems in analysis II.* Springer-Verlag, 1976.

[6] Rigler, A.K., Trimble, S.Y., and Varga, R.S. *Sharp lower bounds for a generalized Jensen inequality.* preprint.