# AN INEQUALITY INVOLVING THE GENERALIZED HYPERGEOMETRIC FUNCTION AND THE ARC LENGTH OF AN ELLIPSE

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**Abstract.** In this paper we verify a conjecture of M. Vuorinen that the Muir approximation is a lower approximation to the arc length of an ellipse. Vuorinen conjectured that $f(x) = {}_2F_1(\frac{1}{2}, -\frac{1}{2}; 1; x) - [(1 + (1 - x)^{3/4})/2]^{2/3}$ is positive for $x \in (0, 1)$. The authors prove a much stronger result which says that the Maclaurin coefficients of $f$ are nonnegative. As a key lemma, we show that ${}_3F_2(-n, a, b; 1 + a + b, 1 + \epsilon - n; 1) > 0$ when $0 < ab/(1 + a + b) < \epsilon < 1$ for all positive integers $n$.

**Key words.** hypergeometric, approximations, elliptical arc length

**AMS subject classifications.** 33C, 41A

**1. Introduction.** Let $a$ and $b$ be the semiaxes of an ellipse with eccentricity $e = \sqrt{a^2 - b^2}/a$. Let $L(a, b)$ denote the arc length of the ellipse. Without loss of generality we can take one of the semiaxes, say $a$, to be 1. Legendre's complete elliptic integral of the 2nd kind can be defined by

$$E(r) = \int_0^{\pi/2} \sqrt{1 - r^2 \sin^2 t}\, dt$$

Elliptic integrals are so named because of their connection with $L(a, b)$. In turn, these are related to Gauss' hypergeometric functions, ${}_2F_1$, defined by

$${}_2F_1(a_1, a_2; b_1; z) = \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n}{(b_1)_n n!} z^n$$

with the Appell (or Pochhammer) symbol $(a)_n = a(a + 1) \cdots (a + n - 1)$ for $n \geq 1$ and $(a)_0 = 1$. We shall need the generalized hypergeometric function, ${}_pF_q$, defined by

$${}_pF_q(a_1, a_2, ..., a_p; b_1, b_2, ..., b_q; z) = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdot (a_2)_n \cdots (a_p)_n}{(b_1)_n \cdot (b_2)_n \cdots (b_q)_n} \frac{z^n}{n!}$$

(see [12], page 73). It was noted by Maclaurin in 1742 (see [2]) that

$$L(1, b) = 4E(e) = 2\pi {}_2F_1(\tfrac{1}{2}, -\tfrac{1}{2}; 1; e^2).$$

There are various references, books and articles, which discuss the relationships between elliptic integrals and hypergeometric functions (see [3], [7]) and their role in applications to physics (see [11], [9]) and in geometric function theory (see [10], [3]). From antiquity several more easily computable approximations to $L(a, b)$ have been suggested. The Almkvist-Berndt survey article [2] has an extensive discussion of these

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approximations. These approximations and their historical and recent connections to the approximations of $\pi$ can be found in the Borweins' book [6]. An excellent source for all of the above ideas is the Anderson-Vamanamurthy-Vuorinen book *Conformal Invarients, Inequalities, and Quasiconformal Mappings* [3].
In 1883, it was proposed by Muir (see [2]) that $L(1,b)$ could be simply approximated by $2\pi[(1+b^{3/2})/2]^{2/3}$. A close numerical examination of the error in this approximation lead M. Vuorinen to pose Problem 5.6 in [13]. This was announced at several international conferences. Letting $x = 1-b^2$, he asked whether the Muir approximation

$$g(x) = \left(\frac{1+(1-x)^{3/4}}{2}\right)^{2/3}$$

is a lower approximation for the value given by the hypergeometric function

$$h(x) = {}_2F_1\left(\tfrac{1}{2}, -\tfrac{1}{2}; 1; x\right),$$

that is, whether

$$h(x) - g(x) \geq 0, \text{ for all } x \in (0,1).$$

We shall prove the following much stronger result:

THEOREM 1.1.
*Let* $g(x) = \sum_{n=0}^{\infty} a_n x^n$ *and* $h(x) = \sum_{n=0}^{\infty} A_n x^n$. *Then,*

$$a_k \leq A_k, \text{ for all } k = 0, 1, 2, \ldots, n, \ldots \tag{1.1}$$

*In particular, the function* $f(x) \equiv [h(x)-g(x)]/x^4$ *is convex and increasing from* $(0,1]$ *onto* $(\alpha, \beta]$, *where* $\alpha = 2^{-14} = 0.000061\cdots$, *and* $\beta = (2/\pi) - 2^{-2/3} = 0.006659\cdots$.
**Remarks** The ideas and techniques used to prove Lemma 2.1 and Theorem 1.1 will be used in [5] to determine surprising hierarchial relationships among the 13 historical approximations to $L(a,b)$ discussed in [2]. These approximations range over four centuries from Kepler's in 1642 to Almkvist's in 1985 and include two from Ramanujan. In addition, the result in Lemma 2.1 has been extended to a wider range of parameters.

**2. Proof of Main Results.** The proof of Theorem 1.1 requires the following Lemma.

LEMMA 2.1. *Suppose* $a, b > 0$. *Then for any* $\epsilon$ *satisfying* $\frac{ab}{1+a+b} < \epsilon < 1$,

$${}_3F_2(-n, a, b; 1+a+b, 1+\epsilon-n; 1) > 0, \quad \textit{for all integers } n \geq 1.$$

For the reader's convenience, we include the following classical identities:
IDENTITY 1. (see [1], page 558, eq. 15.2.24) If $|z| < 1$,

$$(c-b-1)\cdot {}_2F_1(a,b;c;z) = (c-1)\cdot {}_2F_1(a,b;c-1;z) - b\cdot \ {}_2F_1(a,b+1;c;z).$$

IDENTITY 2. (see [12], page 60, Thm. 21) If $|z| < 1$,

$${}_2F_1(a,b;c;z) = (1-z)^{c-a-b}\cdot {}_2F_1(c-a,c-b;c;z).$$

IDENTITY 3. (see [8], page 59, eq. 3.1.1) If $F = {}_3F_2$,

$$F(-n, a, b; c, d; 1) = \frac{(d-b)_n}{(d)_n} F(-n, c-a, b; c, 1+b-d-n; 1).$$

IDENTITY 4. (see [12], page 82, eq. 14) If $F = {}_3F_2$ and $|z| < 1$,

$$\begin{aligned}(a_1 - a_2) \cdot F(a_1, a_2, a_3; b_1, b_2; z) \\ = a_1 \cdot F(a_1+1, a_2, a_3; b_1, b_2; z) - a_2 \cdot F(a_1, a_2+1, a_3; b_1, b_2; z).\end{aligned}$$

**Proof of Lemma 2.1:** Using an idea suggested in [4] we let $F = {}_3F_2$ and consider the generating function

$$f(r) = \sum_{n=0}^{\infty} \frac{-(-\epsilon)_n}{n!} F(-n, a, b; 1+a+b, 1+\epsilon-n; 1) r^n = \sum_{n=0}^{\infty} c_n r^n,$$

where $|r| < 1$. Note that $-(-\epsilon)_n > 0$ for $0 < \epsilon < 1$ and for all $n \geq 1$. Thus we seek to verify that $c_n > 0$ for all $n \geq 1$.
In this direction, we have

$$\begin{aligned}
f(r) &= \sum_{n=0}^{\infty} \frac{-(-\epsilon)_n}{n!} \sum_{k=0}^{n} \frac{(-n)_k (a)_k (b)_k}{(a+b+1)_k (1+\epsilon-n)_k k!} r^n \\
&= \sum_{n=0}^{\infty} \frac{-(-\epsilon)_n}{(1)_n} \sum_{k=0}^{n} \frac{\frac{(-1)^k (1)_n}{(1)_{n-k}} (a)_k (b)_k}{(a+b+1)_k \frac{(-1)^k (-\epsilon)_n}{(-\epsilon)_{n-k}} k!} r^n \\
&\quad \left\{ \text{using } (\alpha)_{n-k} = \tfrac{(-1)^k (\alpha)_n}{(1-\alpha-n)_k} \text{ and } (1)_n = n! \right\} \\
&= -\sum_{n=0}^{\infty} \sum_{k=0}^{n} \left( \frac{(a)_k (b)_k}{(a+b+1)_k k!} r^k \right) \left( \frac{(-\epsilon)_{n-k}}{(n-k)!} r^{n-k} \right) \\
&= -\sum_{n=0}^{\infty} \left( \frac{(-\epsilon)_n}{(n)!} r^n \right) \sum_{k=0}^{\infty} \left( \frac{(a)_k (b)_k}{(a+b+1)_k k!} r^k \right) \qquad \text{(see [12], page 57, eq. 2)} \\
&= -(1-r)^{\epsilon} \; {}_2F_1(a, b; a+b+1; r).
\end{aligned}$$

Differentiating, we have

$$\begin{aligned}
f'(r) &= \epsilon (1-r)^{\epsilon-1} \; {}_2F_1(a, b; a+b+1; r) \\
&\quad - \frac{ab(1-r)^{\epsilon}}{(a+b+1)} \; {}_2F_1(a+1, b+1; a+b+2; r).
\end{aligned} \tag{2.1}$$

An application of IDENTITY 1 followed by IDENTITY 2 to ${}_2F_1(a+1, b+1; a+b+2; r)$ yields

$$\begin{aligned}
&\frac{ab(1-r)^{\epsilon}}{(a+b+1)} {}_2F_1(a+1, b+1; a+b+2; r) \\
&\quad = \frac{b(1-r)^{\epsilon}}{(a+b+1)} [(a+b+1) \cdot {}_2F_1(a+1, b+1; a+b+1; r) - \\
&\qquad (b+1) \cdot {}_2F_1(a+1, b+2; a+b+2; r)] \\
&\quad = (1-r)^{\epsilon-1} \left[ b \cdot {}_2F_1(a, b; a+b+1; r) - \frac{b(b+1)}{(a+b+1)} \cdot {}_2F_1(a, b+1; a+b+2; r) \right].
\end{aligned}$$

Thus (2.1) becomes

$$
\begin{aligned}
f'(r) &= (1-r)^{\epsilon-1}\left[(\epsilon-b)\cdot {}_2F_1(a,b;a+b+1;r)+\right.\\
&\qquad \left.\frac{b(b+1)}{(a+b+1)}\cdot {}_2F_1(a,b+1;a+b+2;r)\right]\\
&= (1-r)^{\epsilon-1}\sum_{n=0}^{\infty}\frac{(a)_n}{n!}\left[\frac{(b)_n(\epsilon-b)}{(a+b+1)_n}+\frac{b(b+1)(b+1)_n}{(a+b+1)(a+b+2)_n}\right]r^n\\
&= (1-r)^{\epsilon-1}\sum_{n=0}^{\infty}\frac{(a)_n}{n!}\left[\frac{(b)_n(\epsilon-b)}{(a+b+1)_n}+\frac{(b+1)(b)_n(b+n)}{(a+b+1)_n(a+b+1+n)}\right]r^n \qquad (2.2)\\
&= (1-r)^{\epsilon-1}\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(a+b+1)_n(a+b+1+n)n!}\times\\
&\qquad [(\epsilon-b)(a+b+1+n)+(b+1)(b+n)]\,r^n\\
&= (1-r)^{\epsilon-1}\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(a+b+1)_n(a+b+1+n)n!}\times \qquad (2.3)\\
&\qquad [\epsilon(a+b+1+n)+n-ab]\,r^n,
\end{aligned}
$$

where (2.2) makes use of $\alpha(\alpha+1)_n = (\alpha)_n(\alpha+n)$. If $\frac{ab}{a+b+1} < \epsilon < 1$, then the expression in (2.3) is the product of two series with all positive Maclaurin series coefficients. Hence $f'$ has all positive Maclaurin series coefficients which is equivalent to the desired result. ∎

Corollary 2.2. *Let* $T_n = {}_3F_2\left(-n,\frac{3}{2},\frac{1}{2};2,\frac{5}{4}-n;1\right)$. *Then for all integers* $n \geq 8$,

$$T_{n+1} > T_n > 0,$$

**Proof of Corollary 2.2:** Let $F = {}_3F_2$ and $B_n = \left(\frac{3}{4}-n\right)_n / \left(\frac{5}{4}-n\right)_n$. Using Identity 3, we have that

$$T_n = B_n F\left(-n,\tfrac{1}{2},\tfrac{1}{2};2,\tfrac{1}{4};1\right).$$

Direct calculation reveals that $T_9 > T_8 > 0 > T_7 > \ldots > T_2 = T_1$. Now suppose that $T_n > T_{n-1} > 0$ for some $n \geq 9$ and note that $B_{n+1}/B_n = \left(n+\frac{1}{4}\right)/\left(n-\frac{1}{4}\right)$. Then

$$
\begin{aligned}
T_{n+1} &= B_{n+1}F\left(-n-1,\tfrac{1}{2},\tfrac{1}{2};2,\tfrac{1}{4};1\right)\\
&= \frac{B_{n+1}}{\left(n+\frac{3}{2}\right)}\left[(n+1)F\left(-n,\tfrac{1}{2},\tfrac{1}{2};2,\tfrac{1}{4};1\right)+\tfrac{1}{2}F\left(-n-1,\tfrac{3}{2},\tfrac{1}{2};2,\tfrac{1}{4};1\right)\right] \qquad (2.4)\\
&= \frac{B_{n+1}(n+1)}{B_n\left(n+\frac{3}{2}\right)}T_n+\frac{B_{n+1}}{2\left(n+\frac{3}{2}\right)}F\left(-n-1,\tfrac{3}{2},\tfrac{1}{2};2,\tfrac{1}{4};1\right)\\
&= \frac{\left(n+\frac{1}{4}\right)(n+1)}{\left(n-\frac{1}{4}\right)\left(n+\frac{3}{2}\right)}T_n+\frac{B_{n+1}}{2\left(n+\frac{3}{2}\right)}F\left(-n-1,\tfrac{3}{2},\tfrac{1}{2};2,\tfrac{1}{4};1\right)\\
&> T_n+\frac{B_{n+1}}{2\left(n+\frac{3}{2}\right)}F\left(-n-1,\tfrac{3}{2},\tfrac{1}{2};2,\tfrac{1}{4};1\right),\\
&\qquad \left\{\text{since } \tfrac{(n+1/4)(n+1)}{(n-1/4)(n+3/2)} > 1 \text{ and } T_n > 0\right\}
\end{aligned}
$$

where (2.4) follows from Identity 4. Since $B_{n+1} < 0$, we shall have that $T_{n+1} > T_n > 0$ provided we show that $F(-n-1, \frac{3}{2}, \frac{1}{2}; 2, \frac{1}{4}; 1) < 0$. To this end, we again apply Identity 3 to observe that

$$F\left(-n-1, \tfrac{3}{2}, \tfrac{1}{2}; 2, \tfrac{1}{4}; 1\right) = \frac{\left(-\frac{1}{4}\right)_{n+1}}{\left(\frac{1}{4}\right)_{n+1}} F\left(-n-1, \tfrac{1}{2}, \tfrac{1}{2}; 2, \tfrac{1}{4}-n; 1\right).$$

Since $\dfrac{\left(-\frac{1}{4}\right)_{n+1}}{\left(\frac{1}{4}\right)_{n+1}} < 0$, we need to show that

$$F\left(-n-1, \tfrac{1}{2}, \tfrac{1}{2}; 2, \tfrac{1}{4}-n; 1\right) > 0.$$

Letting $m = n+1$, $a = b = \frac{1}{2}$, and $\epsilon = \frac{1}{4}$, it follows from Lemma 2.1 that

$$F\left(-n-1, \tfrac{1}{2}, \tfrac{1}{2}; 2, \tfrac{1}{4}-n; 1\right) = F(-m, a, b; a+b+1, 1+\epsilon-m; 1) > 0.$$

Hence $T_{n+1} > T_n > 0$ for all integers $n \geq 8$ by induction. ∎

**Proof of Theorem 1.1.** Clearly,

$$A_n = \frac{\left(\frac{1}{2}\right)_n \left(-\frac{1}{2}\right)_n}{n!n!}.$$

Computing the logarithmic derivative of $g$ we have

$$\frac{g'(x)}{g(x)} = -\frac{1}{2}\left(\frac{(1-x)^{-\frac{1}{4}}}{1+(1-x)^{\frac{3}{4}}}\right)$$

which implies

$$\left(\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n\right)\left((1-x)^{\frac{1}{4}}+1-x\right) = -\frac{1}{2}\sum_{n=0}^{\infty}a_n x^n. \tag{2.5}$$

The coefficients of $x^n$ of the left-hand side of (2.5) are obtained from the Cauchy product of the two terms. Solving for $a_{n+1}$ yields (by extracting the $n^{th}$ and $(n-1)^{st}$ terms from the Cauchy product)

$$a_{n+1} = \frac{1}{2(n+1)}\left[\left(\frac{5}{4}n - \frac{1}{2}\right)a_n - \sum_{k=0}^{n-2}(k+1)a_{k+1}\frac{\left(-\frac{1}{4}\right)_{n-k}}{(n-k)!}\right]. \tag{2.6}$$

We now verify (1.1) using an inductive argument. Clearly, the coefficients of the terms $a_k$ in (2.6) are nonnegative. Computation gives: $a_0 = A_0 = 1,\ a_1 = A_1 = -1/4,\ a_2 = A_2 = -3/64,\ a_3 = A_3 = -5/2^8,\ a_4 = -11/2^{10}$ and $A_4 = -175/2^{14}$. Suppose that the inequality in (1.1) holds for $4 \leq k \leq n$. We have from (2.6)

(2.7) $a_{n+1} \leq$

$$\frac{1}{2(n+1)}\left[\left(\frac{5}{4}n - \frac{1}{2}\right)\frac{\left(\frac{1}{2}\right)_n\left(-\frac{1}{2}\right)_n}{n!n!} - \sum_{k=0}^{n-2}(k+1)\frac{\left(\frac{1}{2}\right)_{k+1}\left(-\frac{1}{2}\right)_{k+1}}{(k+1)!(k+1)!}\frac{\left(-\frac{1}{4}\right)_{n-k}}{(n-k)!}\right].$$

We need to show that the right-hand side of (2.7) is less than or equal to $A_{n+1} = \frac{\left(\frac{1}{2}\right)_{n+1}\left(-\frac{1}{2}\right)_{n+1}}{(n+1)!(n+1)!}$, that is,

$$\left(\frac{5}{4}n-\frac{1}{2}\right)\frac{\left(\frac{1}{2}\right)_n\left(-\frac{1}{2}\right)_n}{n!n!}-2(n+1)\frac{\left(\frac{1}{2}\right)_{n+1}\left(-\frac{1}{2}\right)_{n+1}}{(n+1)!(n+1)!}\leq \sum_{k=0}^{n-2}(k+1)\frac{\left(\frac{1}{2}\right)_{k+1}\left(-\frac{1}{2}\right)_{k+1}}{(k+1)!(k+1)!}\frac{\left(-\frac{1}{4}\right)_{n-k}}{(n-k)!}. \tag{2.8}$$

After adding the $(n-1)^{st}$ and $n^{th}$ terms of the right-hand side of (2.8) to inequality (2.8) and then simplifying, we use $(a)_{k+1}=(a+k)(a)_k$, $(a)_{n-k}=(-1)^k(a)_k/(1-a-n)_k$, the fact that $(-n)_k=0$ for $k\geq n+1$, and the definition of ${}_3F_2$ to obtain

$$\frac{\left(\frac{1}{2}\right)_n\left(-\frac{1}{2}\right)_n}{n!n!}\cdot\frac{(2n-1)}{4(n+1)}\leq -\left(\frac{1}{4}\right)\left(-\frac{1}{4}\right)_n\frac{{}_3F_2\left(-n,\frac{1}{2},\frac{3}{2};2,\frac{5}{4}-n;1\right)}{n!},$$

or equivalently

$${}_3F_2\left(-n,\tfrac{1}{2},\tfrac{3}{2};2,\tfrac{5}{4}-n;1\right)\geq\frac{\left(\frac{1}{2}\right)_n^2}{\left(-\frac{1}{4}\right)_n(n+1)!}. \tag{2.9}$$

Clearly, the right-hand side of (2.9) is negative for all $n\geq 1$. Inequality (2.9) can be explicitly verified for $0\leq n\leq 7$. For $n\geq 8$, inequality (2.9) follows from Corollary 2.2. Thus, the inequality in (1.1) also holds for $k=n+1$. Hence, by induction (1.1) holds for all $k\in\mathbb{N}\cup\{0\}$.

Finally, the convexity and monotonicity of $f$ are clear. By, l'Hôpital's Rule, $f(0^+)=A_4-a_4=1/2^{14}=1/16384$, while the value of $f(1)$ is clear. ∎

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## REFERENCES

[1] Abramowitz M. and Stegun I., Eds., *Handbook of Mathematical Functions* (1965) Dover, New York.

[2] Almkvist G. and Berndt B., "Gauss, Landen, Ramanujan, the arithmetic-geometric mean, ellipses, pi and the Ladies Diary," Amer. Math. Monthly **95** (1988) 585-608.

[3] Anderson, G.D., Vamanamurthy, M.K. and Vuorinen, M., *Conformal Invarients, Inequalities, and Quasiconformal Mappings* (1997) J. Wiley & Sons, New York.

[4] Askey R., Gasper G. and Ismail M., "A positive sum from summability theory," J. of Approx. Theory **13** (1975) 413-420.

[5] Barnard, R.W., Pearce, K. and Richards, K.C., "A Monotonicity Property Involving ${}_3F_2$ and Comparisons of Classical Approximations of Elliptical Arc Length", (preprint).

[6] Borwein, J.M. and Borwein, P.B., *Pi and the AGM — A Study in Analytic Number Theory and Computational Complexity* (1987) J. Wiley & Sons, New York.

[7] Carlson, B.C., *Special Functions of Applied Mathematics* (1977) Academic Press, New York.

[8] Gasper G. and Rahman M., *Basic Hypergeometric Series* (1990) Cambridge University Press, Cambridge.

[9] Greenhill, A.G., *The Applications of Elliptic Functions* (1954) Dover, New York.

[10] Henrici, P., *Applied and Computational Complex Analysis*, Vol. I, (1974) J. Wiley & Sons, New York.

[11] Lawden, D.F., *Ellipitic Functions and Applications* Applied Math. Sciences **80** (1989) Springer-Verlag, New York.

[12] Rainville E., *Special Functions* (1960) Macmillan, New York.

[13] Vuorinen M., "Hypergeometric Functions in Geometric Function Theory," in the *Proceedings of Special Functions and Differential Equations*, Eds.: Srinivasa K.R., Jagannthan R. and Van der Jeugy G., (1998) Allied Publishers, to be published.