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A MONOTONICITY PROPERTY INVOLVING $_3F_2$ AND COMPARISONS OF THE CLASSICAL APPROXIMATIONS OF ELLIPTICAL ARC LENGTH*

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Abstract. Conditions are determined under which $_3F_2\left(-n,a,b;a+b+2,\varepsilon-n+1;1\right)$ is a monotone function of n satisfying $ab\cdot {}_3F_2\left(-n,a,b;a+b+2,\varepsilon-n+1;1\right) \geq ab\cdot {}_2F_1\left(a,b;a+b+2;1\right)$. Motivated by a conjecture of Vuorinen [*Proceedings of Special Functions and Differential Equations*, K. S. Rao, R. Jagannathan, G. Vanden Berghe, J. Van der Jeugt, eds., Allied Publishers, New Delhi, 1998], the corollary that $_3F_2(-n,-\frac{1}{2},-\frac{1}{2};1,\varepsilon-n+1;1) \geq \frac{4}{\pi}$, for $1 > \epsilon \geq \frac{1}{4}$ and $n \geq 2$, is used to determine surprising hierarchical relationships among the 13 known historical approximations of the arc length of an ellipse. This complete list of inequalities compares the Maclaurin series coefficients of $_2F_1$ with the coefficients of each of the known approximations, for which maximum errors can then be established. These approximations range over four centuries from Kepler's in 1609 to Almkvist's in 1985 and include two from Ramanujan.

Key words. hypergeometric, approximations, elliptical arc length

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1. Introduction. Let $L(x,y)$ be the arc length of an ellipse with semiaxes of length x and y (with $x \geq y > 0$) and let $\lambda \equiv \dfrac{x-y}{x+y}$. In 1742, Maclaurin [12] determined that

$$L(x,y) = \pi(x+y)\,\cdot\,{}_2F_1\left(-\frac{1}{2},-\frac{1}{2};1;\lambda^2\right), \tag{1}$$

where $_2F_1$ is the hypergeometric function defined by

$$_2F_1\left(a,b;c;z\right) \equiv 1 + \sum_{n=1}^{\infty} \frac{(a)_n(b)_n z^n}{(c)_n n!}$$

with the Appell (or Pochhammer) symbol $(a)_n \equiv a(a+1)\cdots(a+n-1)$ for $n \geq 1$ and $(a)_0 \equiv 1$, $a \neq 0$. (For more background information, see [2], [14], [9], and the recent survey article [8] by the first author.) In [2], Almkvist and Berndt compiled and presented the list of the approximations in Table 1.1 for

$$G(\lambda) \equiv {}_2F_1\left(-\frac{1}{2},-\frac{1}{2};1;\lambda^2\right) = \frac{L(x,y)}{\pi(x+y)}.$$

These approximations and their historical and recent connections to the approximations of π can be found in the Borweins' book [10]. Another excellent source for historical and current studies of these topics is the book [5] by Anderson, Vamanamurthy, and Vuorinen.

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TABLE 1.1
Approximations of $G(\lambda) \equiv {}_2F_1(-\frac{1}{2}, -\frac{1}{2}; 1; \lambda^2)$ (see [2]).

Discoverer(s) and year of discovery	Approximation $A_p(\lambda)$	δ_p = first nonzero term in the Maclaurin series for $\Delta_p(\lambda) \equiv A_p(\lambda) - G(\lambda)$
Kepler, 1609	$A_1(\lambda) \equiv (1-\lambda^2)^{1/2}$	$\delta_1 = -\frac{3}{4}\lambda^2$
Euler, 1773	$A_2(\lambda) \equiv (1+\lambda^2)^{1/2}$	$\delta_2 = \frac{1}{4}\lambda^2$
Sipos, 1792 Ekwall, 1973	$A_3(\lambda) \equiv \dfrac{2}{1+\sqrt{1-\lambda^2}}$	$\delta_3 = \frac{7}{64}\lambda^4$
Peano, 1889	$A_4(\lambda) \equiv \frac{3}{2} - \frac{1}{2}(1-\lambda^2)^{1/2}$	$\delta_4 = \frac{3}{64}\lambda^4$
Muir, 1883	$A_5(\lambda) \equiv \left(\dfrac{(1+\lambda)^{3/2}+(1-\lambda)^{3/2}}{2}\right)^{2/3}$	$\delta_5 = -\frac{1}{64}\lambda^4$
Lindner, 1904-1920 Nyvoll, 1978	$A_6(\lambda) \equiv \left(1+\dfrac{\lambda^2}{8}\right)^2$	$\delta_6 = -\frac{1}{2^8}\lambda^6$
Selmer, 1975	$A_7(\lambda) \equiv 1 + \dfrac{\lambda^2/4}{1-\lambda^2/16}$	$\delta_7 = -\frac{3}{2^{10}}\lambda^6$
Ramanujan, 1914 Fergestad, 1951	$A_8(\lambda) \equiv 3 - \sqrt{4-\lambda^2}$	$\delta_8 = -\frac{1}{2^9}\lambda^6$
Almkvist, 1978	$A_9(\lambda) \equiv 2\dfrac{\left(1+\sqrt{1-\lambda^2}\right)^2 + \lambda^2\sqrt{1-\lambda^2}}{\left(1+\sqrt{1-\lambda^2}\right)\left(1+\sqrt[4]{1-\lambda^2}\right)^2}$	$\delta_9 = \frac{15}{2^{14}}\lambda^8$
Bronshtein and Semendyayev, 1964 Selmer, 1975	$A_{10}(\lambda) \equiv \dfrac{64-3\lambda^4}{64-16\lambda^2}$	$\delta_{10} = -\frac{9}{2^{14}}\lambda^8$
Selmer, 1975	$A_{11}(\lambda) \equiv \frac{3}{2} + \frac{\lambda^2}{8} - \frac{1}{2}\left(1-\dfrac{\lambda^2}{2}\right)^{1/2}$	$\delta_{11} = -\frac{5}{2^{14}}\lambda^8$
Jacobsen and Waadeland, 1985	$A_{12}(\lambda) \equiv \dfrac{256-48\lambda^2-21\lambda^4}{256-112\lambda^2+3\lambda^4}$	$\delta_{12} = -\frac{33}{2^{18}}\lambda^{10}$
Ramanujan, 1914	$A_{13}(\lambda) \equiv 1 + \dfrac{3\lambda^2}{10+\sqrt{4-3\lambda^2}}$	$\delta_{13} = -\frac{3}{2^{17}}\lambda^{10}$

Recently, several inequalities between various mean values and the hypergeometric function were proved in [10], [15], and the dependence of the hypergeometric function ${}_2F_1(a, b; c; z)$ on its parameters was studied in [4], [6]. These results led to a conjecture of Vuorinen (see [16]) concerning Muir's approximation A_5. Vuorinen conjectured (see [16]) that

$$A_5(\lambda) \leq G(\lambda) \quad \text{for all } \lambda \in [0,1]. \tag{2}$$

That is, Vuorinen conjectured that A_5 is a *lower bound* for G. This conjecture was recently proved by the authors in [9] which has become the genesis of the present article. Moreover, the results here attest to the adage that a single conjecture may have many ramifications. Also, note that A_5 is one of the mean values studied in [15]. More approximations for hypergeometric functions in terms of such mean values are actively being sought. For example, let $\nu \in \mathbf{R}\backslash\{0\}$ and define

$$M_\nu(\lambda) \equiv \left[\frac{(1+\lambda)^\nu + (1-\lambda)^\nu}{2}\right]^{1/\nu}.$$

H. Alzer [3] originally made the following conjecture.

CONJECTURE. *The inequalities*

$$M_\alpha(\lambda) \leq G(\lambda) \leq M_\beta(\lambda) \quad \textit{hold for all } \lambda \in (0,1) \tag{3}$$

if and only if

$$\alpha \leq 3/2 \quad \textit{and} \quad \beta \geq (\ln 2)/\left(\ln \frac{\pi}{2}\right) \approx 1.53.$$

As noted by Alzer [3], it follows from our results (see the set of inequalities in expression (4)) that (3) holds with $\alpha = 3/2$ and $\beta = 2$. Moreover, for a fixed λ, $M_\nu(\lambda)$ is an increasing function of ν. Thus it follows that (3) holds for all $\alpha \leq 3/2$ and $\beta \geq 2$. It can be shown that $\alpha = 3/2$ is sharp.

2. Main results. In an earlier paper (see [9]), the authors were able to verify inequality (2) by working with the original version of Vuorinen's conjecture in terms of the eccentricity (see (5) and (6)). In this direction, a generating function argument (motivated by [7]) was used to obtain the following general result (which will also be applied in this paper to obtain Theorem 2.5).

THEOREM 2.1 (see [9]). *Suppose* $a, b > 0$. *Then for any* ε *satisfying* $1 > \varepsilon \geq \frac{ab}{a+b+1}$, *it follows that*

$${}_3F_2\left(-n, a, b; a+b+1, \varepsilon - n + 1; 1\right) \geq 0,$$

for all integers $n \geq 1$, *where* ${}_3F_2$ *is the generalized hypergeometric function.*

In light of the conjecture in (2), the following question naturally arises:

> *Which of the remaining approximations given in Table* 1.1 *are upper bounds or lower bounds for* G?

An attempt to compare an approximation A_p with G motivates an analysis of the term δ_p (the first nonzero term in the Maclaurin series representation for the error function $\Delta_p(\lambda) \equiv A_p(\lambda) - G(\lambda)$). What information does δ_p provide? Certainly the leading term can be viewed as a measure of accuracy of the given approximation, and the error function $\Delta_p(\lambda)$ will have the same sign as δ_p for *sufficiently small* λ. For example, $\delta_1 < 0$ and it follows directly that A_1 is a lower bound for G, as Kepler intended (see [2, p. 599]). In this case, the sign of δ_1 is indicative of the sign of $\Delta_1(\lambda)$ *for all* $\lambda \in [0, 1]$. Almkvist and Berndt proved (see [2, p. 603]) that Ramanujan's first estimate A_8 is a lower bound for G by proving the significantly stronger result that the nonzero Maclaurin series coefficients of Δ_8 all have the same (negative) sign. A numerical investigation suggests that a similar trait might be shared by other approximations given in Table 1.1. In this article, it will be shown that all of the approximations given in Table 1.1 satisfy the following property:

> *The sign of the error function* $\Delta_p(\lambda)$ *coincides with the sign of the leading term* δ_p <u>*for all* $\lambda \in [0, 1]$</u>.

Moreover, for all but two of the approximations, it will be established that the nonzero Maclaurin series coefficients of Δ_p all have the same sign as δ_p. (Only Euler's approximation and Muir's approximation fail to satisfy this condition.) As a consequence of the forthcoming results, each function $|\Delta_p|$ is a strictly increasing function of λ, for $p = 1, \ldots, 13$. Therefore, $0 = |\Delta_p(0)| < |\Delta_p(\lambda)| < |\Delta_p(1)|$ for all $\lambda \in (0, 1)$. For example, the maximum error for Ramanujan's second estimate is $|\Delta_{13}(1)| = |\frac{14}{11} - \frac{4}{\pi}| \approx 0.000512$ and satisfies $|\Delta_{13}(1)| < |\Delta_p(1)|$ for $p = 1, \ldots, 12$. In this direction, we will prove the following three propositions.

PROPOSITION 2.2. *Let* $G(\lambda) \equiv \sum_{n=0}^{\infty} \alpha_n \lambda^{2n}$ *and* $A_p(\lambda) \equiv \sum_{n=0}^{\infty} \beta_n^{(p)} \lambda^{2n}$ *where* $\alpha_n \equiv (\frac{(-1/2)_n}{n!})^2$ *and each* A_p *is defined as in Table* 1.1. *Then*

$$\beta_n^{(12)} \leq \alpha_n \leq \beta_n^{(9)} \quad \textit{for all integers } n \geq 0.$$

Therefore, the error functions $|\Delta_9|$ *and* $|\Delta_{12}|$ *are strictly increasing and*

$$A_{12}(\lambda) \leq G(\lambda) \leq A_9(\lambda) \quad \text{for all } \lambda \in [0,1].$$

PROPOSITION 2.3. *Let* $G(\lambda) \equiv \sum_{n=0}^{\infty} \alpha_n \lambda^{2n}$ *and* $A_p(\lambda) \equiv \sum_{n=0}^{\infty} \beta_n^{(p)} \lambda^{2n}$ *where* $\alpha_n \equiv (\frac{(-1/2)_n}{n!})^2$ *and each* A_p *is defined as in Table* 1.1. *Then*

$$\beta_n^{(1)} \leq \beta_n^{(6)} \leq \beta_n^{(7)} \leq \beta_n^{(8)} \leq \beta_n^{(10)} \leq \beta_n^{(11)} \leq \beta_n^{(13)} \leq \alpha_n \leq \beta_n^{(4)} \leq \beta_n^{(3)}$$

for all integers $n \geq 0$. *Therefore, the corresponding error functions* $|\Delta_p|$ *are strictly increasing and*

$$A_1(\lambda) \leq A_6(\lambda) \leq A_7(\lambda) \leq A_8(\lambda) \leq A_{10}(\lambda) \leq A_{11}(\lambda) \leq A_{13}(\lambda) \leq G(\lambda) \leq A_4(\lambda) \leq A_3(\lambda)$$

for all $\lambda \in [0,1]$.

The next proposition addresses the two remaining estimates: Euler's approximation A_2 and Muir's approximation A_5. The claim will be made that

$$A_5(\lambda) \equiv \left(\frac{(1+\lambda)^{3/2} + (1-\lambda)^{3/2}}{2}\right)^{2/3} \leq G(\lambda) \leq (1+\lambda^2)^{1/2} \equiv A_2(\lambda) \tag{4}$$

for all $\lambda \in$ [0,1]. As we have noted, the nonzero Maclaurin series coefficients of Δ_2 and Δ_5 (as functions of λ) do not have constant sign. In order to verify the inequalities in (4), we make use of the known fact due to Landen and Ivory (e.g., see [2, p. 598]) that

$$G(\lambda) \equiv {}_2F_1\left(-\frac{1}{2}, -\frac{1}{2}; 1; \lambda^2\right) = \frac{2x}{x+y} \cdot {}_2F_1\left(\frac{1}{2}, -\frac{1}{2}; 1; \xi^2\right), \tag{5}$$

where $\lambda \equiv (x-y)/(x+y)$ and $\xi \equiv (1/x)\sqrt{x^2-y^2}$ is the eccentricity of the original ellipse (see (1)). Without loss of generality, assume that $1 = x \geq y \geq 0$. A change of variable from λ to ξ can be accomplished in (4) by using (5) and the substitutions $\lambda = (1-y)/(1+y)$ and $y = \sqrt{1-\xi^2}$. Multiplying through by $(1+y)/2$ and simplifying, we see that the inequalities in (4) are equivalent to

$$\left(\frac{1+(1-\xi^2)^{3/4}}{2}\right)^{2/3} \leq {}_2F_1\left(\frac{1}{2}, -\frac{1}{2}; 1; \xi^2\right) \leq (1-\xi^2/2)^{1/2} \tag{6}$$

for all $\xi \in$ [0,1]. (The first inequality in (6) is the original version of Vuorinen's conjecture [16].)

It is interesting to note that one can show that the functions in (6) can be shown to satisfy the stated inequalities by establishing that the coefficients of their respective Maclaurin series, *expanded in powers of* ξ, satisfy the corresponding inequality relationships. In view of the preceding discussion, we now state the following proposition.

PROPOSITION 2.4 (see [9]). *Let* G *and* A_p *be as defined in Table* 1.1 *and let*

$$1 + \sum_{n=1}^{\infty} b_n \xi^{2n} \equiv \left(\frac{1+(1-\xi^2)^{3/4}}{2}\right)^{2/3} \quad \textit{and} \tag{7}$$

$$1 + \sum_{n=1}^{\infty} c_n \xi^{2n} \equiv (1-\xi^2/2)^{1/2}. \tag{8}$$

It follows that

$$b_n \le \frac{(1/2)_n(-1/2)_n}{n!\cdot n!} \le c_n \quad \textit{for all integers } n \ge 1.$$

Therefore, (6) *holds and is equivalent to* $A_5(\lambda) \le G(\lambda) \le A_2(\lambda)$ *for all* $\lambda \in [0,1]$.

Remark. If we apply the identity in (5) with $\lambda = (1-\sqrt{1-\xi^2})/(1+\sqrt{1-\xi^2})$, the definition of A_2, and simplify, we obtain $\Delta_2(\lambda) = 2[(1-\xi^2/2)^{1/2} - {}_2F_1(\frac{1}{2},-\frac{1}{2};1;\xi^2)]/(1+\sqrt{1-\xi^2})$. Proposition 2.4 implies that $(1-\xi^2/2)^{1/2} - {}_2F_1(\frac{1}{2},-\frac{1}{2};1;\xi^2)$ is a strictly increasing function of ξ. Therefore $\Delta_2(\lambda)$ is a strictly increasing function of ξ. Since $\xi = \frac{2\sqrt{\lambda}}{1+\lambda}$ is a strictly increasing function of λ on [0,1], it follows that $|\Delta_2|$ is a strictly increasing function of λ. A similar argument can be applied to $|\Delta_5|$.

Although some of the inequalities in the above propositions are straightforward, several proved to be surprisingly challenging to verify. In particular, the effort involving Almkvist's approximation A_9 precipitated the discovery of some deeper results involving the generalized hypergeometric function ${}_3F_2$, which are also of independent interest. In this direction, our main general results are as follows.

THEOREM 2.5. *Let* $1 > a \ge b > -1$ *and* $1 > \varepsilon \ge \frac{(a+1)(b+2)}{a+b+4}$. *Then* $T_n \equiv {}_3F_2(-n,a,b;a+b+2,\varepsilon-n+1;1)$ *satisfies*

$$ab(T_n - T_{n+1}) \ge 0 \quad \text{for all integers } n \ge 2.$$

COROLLARY 2.6. *Let* $1 > a \ge b > -1$ *and* $1 > \varepsilon \ge \frac{(a+1)(b+2)}{a+b+4}$. *Then* $T_n \equiv {}_3F_2(-n,a,b;a+b+2,\varepsilon-n+1;1)$ *satisfies*

$$abT_n \ge abT_{n+1} \ge ab\cdot {}_2F_1(a,b;a+b+2;1) \quad \text{for all integers } n \ge 2.$$

COROLLARY 2.7. *Let* $1 > \varepsilon \ge \frac{1}{4}$. *Then* $T_n \equiv {}_3F_2\left(-n,-\frac{1}{2},-\frac{1}{2};1,\varepsilon-n+1;1\right)$ *satisfies*

$$T_n \ge T_{n+1} \ge \frac{4}{\pi} \quad \text{for all integers } n \ge 2.$$

3. Verification of coefficient inequalities.

Proof of Proposition 2.2. *Part* I: *Almkvist's Approximation* A_9. Let $s \equiv (1-\lambda^2)^{1/2}$ and $\beta_n \equiv \beta_n^{(9)}$. It follows that

$$A_9(\lambda) = 2\left[\frac{(1+s)+(1-s)s}{(1+\sqrt{s})^2}\right] = \sum_{n=0}^{\infty}\beta_n\lambda^{2n},$$

which implies that

$$2(1+2s-s^2) = (1+2\sqrt{s}+s)\sum_{n=0}^{\infty}\beta_n\lambda^{2n}. \tag{9}$$

By replacing s by $(1-\lambda^2)^{1/2}$ and applying $(1-\lambda^2)^q = \sum_{n=0}^{\infty}\frac{(-q)_n}{n!}\lambda^{2n}$, we may change (9) to the form

$$\begin{aligned}&2\lambda^2 + 4\sum_{n=0}^{\infty}\frac{(-1/2)_n}{n!}\lambda^{2n}\\ &\quad = \sum_{n=0}^{\infty}\beta_n\lambda^{2n} + 2\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(-1/4)_{n-k}}{(n-k)!}\beta_k\lambda^{2n} + \sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(-1/2)_{n-k}}{(n-k)!}\beta_k\lambda^{2n}.\end{aligned}$$

Equating the coefficients of λ^{2n}, we obtain $\beta_0 = 1$, $\beta_1 = 1/4$, and

$$4\frac{(-1/2)_n}{n!} = \beta_n + 2\sum_{k=0}^{n} \frac{(-1/4)_{n-k}}{(n-k)!}\beta_k + \sum_{k=0}^{n} \frac{(-1/2)_{n-k}}{(n-k)!}\beta_k \quad \text{for } n \geq 2.$$

Solving for β_n, we have the recursive relationship

$$\beta_n = \frac{(-1/2)_n}{n!} - \frac{1}{2}\sum_{k=0}^{n-1} \frac{(-1/4)_{n-k}}{(n-k)!}\beta_k - \frac{1}{4}\sum_{k=0}^{n-1} \frac{(-1/2)_{n-k}}{(n-k)!}\beta_k \quad \text{for } n \geq 2. \tag{10}$$

We will use (10) and induction to show that

$$\beta_n \geq \alpha_n \quad \text{for all } n \geq 0. \tag{11}$$

First note that $\beta_n = \alpha_n$ for $n = 0, 1, 2$. Now let $n \geq 2$ and suppose that $\beta_k \geq \alpha_k$ for all $k = 0, \ldots, n-1$. Since the coefficients of β_k in (10) are all positive, it follows that

$$\beta_n \geq \frac{(-1/2)_n}{n!} - \frac{1}{2}\sum_{k=0}^{n-1} \frac{(-1/4)_{n-k}}{(n-k)!}\alpha_k - \frac{1}{4}\sum_{k=0}^{n-1} \frac{(-1/2)_{n-k}}{(n-k)!}\alpha_k.$$

Thus (11) will be established if we can verify that

$$\frac{(-1/2)_n}{n!} - \frac{1}{2}\sum_{k=0}^{n-1} \frac{(-1/4)_{n-k}}{(n-k)!}\alpha_k - \frac{1}{4}\sum_{k=0}^{n-1} \frac{(-1/2)_{n-k}}{(n-k)!}\alpha_k \geq \alpha_n \quad \text{for } n \geq 2. \tag{12}$$

Next we use the identities $(c)_{n-k} = \frac{(-1)^k (c)_n}{(1-c-n)_k}$ and $(1)_n = n!$ and add the corresponding nth term of each summation to both sides. Then (12) becomes

$$\frac{(-1/2)_n}{n!} - \frac{(-1/4)_n}{2 \cdot n!}\sum_{k=0}^{n} \frac{(-n)_k}{(5/4-n)_k}\alpha_k - \frac{(-1/2)_n}{4 \cdot n!}\sum_{k=0}^{n} \frac{(-n)_k}{(3/2-n)_k}\alpha_k \geq \frac{\alpha_n}{4}. \tag{13}$$

Now we apply $\alpha_k \equiv \left(\frac{(-1/2)_k}{k!}\right)^2$ and the definition of ${}_3F_2$, then divide both sides of (13) by $\frac{-(-1/2)_n}{4 \cdot n!}$, and simplify. Then inequality (13) becomes

$$\begin{aligned} P(n) \cdot {}_3F_2&\left(-n, -\frac{1}{2}, -\frac{1}{2}; 1, \frac{5}{4} - n; 1\right) \\ &+ {}_3F_2\left(-n, -\frac{1}{2}, -\frac{1}{2}; 1, \frac{3}{2} - n; 1\right) \geq Q(n), \end{aligned} \tag{14}$$

where $P(n) \equiv 2\frac{(-1/4)_n}{(-1/2)_n}$ and $Q(n) \equiv 4 - \frac{(-1/2)_n}{n!}$. For $n \geq 2$, these can be shown to satisfy

$$P(n) \leq P(n+1) \quad \text{and} \tag{15}$$

$$Q(n) \geq Q(n+1). \tag{16}$$

We first note that inequality (14) can be confirmed directly for $n = 2, \ldots, 6$. An application of Corollary 2.7 (to be proved in the following section), with the respective values of $\varepsilon = 1/4$ and $\varepsilon = 1/2$, yields

$${}_3F_2\left(-n, -\frac{1}{2}, -\frac{1}{2}; 1, \frac{5}{4} - n; 1\right) \geq \frac{4}{\pi} \quad \text{and} \tag{17}$$

$${}_3F_2\left(-n, -\frac{1}{2}, -\frac{1}{2}; 1, \frac{3}{2} - n; 1\right) \geq \frac{4}{\pi} \tag{18}$$

for all $n \geq 2$. From inequalities (15)–(18) with $n \geq 6$, it follows that

$$P(n) \cdot {}_3F_2\left(-n, -\frac{1}{2}, -\frac{1}{2}; 1, \frac{5}{4} - n; 1\right) + {}_3F_2\left(-n, -\frac{1}{2}, -\frac{1}{2}; 1, \frac{3}{2} - n; 1\right)$$
$$\geq P(6)\frac{4}{\pi} + \frac{4}{\pi} \geq Q(6) \geq Q(n).$$

Therefore, inequality (14) holds for all $n \geq 2$ and hence $\beta_n^{(9)} \equiv \beta_n \geq \alpha_n$ for all $n \geq 0$. That is, Almkvist's approximation satisfies the property that all of the nonzero Maclaurin series coefficients of Δ_9 are positive. This concludes the proof of Part I of Proposition 2.2.

Proof of Proposition 2.2. *Part* II: *Jacobsen and Waadeland's Approximation* A_{12}. Now we seek to show that the approximation A_{12} satisfies the property that all of the nonzero Maclaurin series coefficients of Δ_{12} are negative. Let $a = 3$, $b = -112$, $c = 256$, and $D = \sqrt{b^2 - 4ac}$. It follows that

$$\frac{1}{au^2 + bu + c} = \frac{2a}{D}\left[\frac{1}{2au + b - D} - \frac{1}{2au + b + D}\right] = \sum_{n=0}^{\infty} d_n u^n \quad \text{for} \quad |u| < \left|\frac{D + b}{2a}\right|,$$

where

$$d_n \equiv \frac{2a}{D}\left[\frac{(-1)^n(2a)^n}{(b - D)^{n+1}} - \frac{(-1)^n(2a)^n}{(b + D)^{n+1}}\right] = \frac{1}{D}\left(\frac{2a}{D - b}\right)^{n+1}\left[\left(\frac{b - D}{b + D}\right)^{n+1} - 1\right].$$

It follows that $d_n > 0$ for all $n \geq 0$ and

$$\begin{aligned} A_{12}(\lambda) &\equiv \frac{256 - 48\lambda^2 - 21\lambda^4}{256 - 112\lambda^2 + 3\lambda^4} \\ &= -7 + \frac{2048 - 832\lambda^2}{256 - 112\lambda^2 + 3\lambda^4} \\ &= -7 + (2048 - 832\lambda^2)\sum_{n=0}^{\infty} d_n \lambda^{2n}. \end{aligned}$$

Now let $\beta_n \equiv \beta_n^{(12)}$. Then the nonzero Maclaurin series coefficients for A_{12} are given by $\beta_0 = 1$ and

$$\beta_n = 2048 d_n - 832 d_{n-1} \quad \text{for all } n \geq 1.$$

Since $(x^{n+1} - 1)/(x - 1) > x$ for $x \equiv (b - D)/(b + D) > 1$, it follows easily that $(2048 d_n)/(832 d_{n-1}) > 1$ for all $n \geq 1$. Thus

$$\beta_n > 0 \quad \text{for all } n \geq 0. \tag{19}$$

Direct calculation reveals that $\beta_n = \alpha_n$ for $n = 0, \ldots, 4$. Also note that

$$(256 - 112\lambda^2 + 3\lambda^4)\sum_{n=0}^{\infty} \beta_n \lambda^{2n} = 256 - 48\lambda^2 - 21\lambda^4.$$

Hence

$$\sum_{n=3}^{\infty} (256\beta_n - 112\beta_{n-1} + 3\beta_{n-2})\lambda^{2n} = 0. \tag{20}$$

Thus the coefficients of λ^{2n} in (20) are zero for all $n \geq 3$. Solving for β_n and using (19), we have

$$\beta_n = (112\beta_{n-1} - 3\beta_{n-2})/256 < \frac{112}{256}\beta_{n-1} \quad \text{for all } n \geq 3.$$

Now suppose that $\beta_n \leq \alpha_n$ for some integer $n \geq 4$, where $\alpha_n \equiv (\frac{(-1/2)_n}{n!})^2$. Then

$$\beta_{n+1} < \frac{112}{256}\beta_n \leq \frac{112}{256}\alpha_n = \frac{112}{256}\frac{\alpha_n}{\alpha_{n+1}}\alpha_{n+1} = \frac{112}{256}\left(\frac{n+1}{n-\frac{1}{2}}\right)^2 \alpha_{n+1} \leq \alpha_{n+1}.$$

Thus $\beta_n^{(12)} \equiv \beta_n \leq \alpha_n$ for all integers $n \geq 0$. This concludes the proof of Part II of Proposition 2.2. $\square$

Before proving Proposition 2.3, we first observe that the nine approximations involved have the following respective Maclaurin series representations (recursive relationships satisfied by $\beta_n^{(13)}$ and $\beta_n^{(3)}$ are developed in the appendix):

$$A_1(\lambda) \equiv (1-\lambda^2)^{1/2} = 1 + \sum_{n=1}^{\infty} \frac{(-1/2)_n}{n!}\lambda^{2n}, \tag{21}$$

$$A_6(\lambda) \equiv \left(1+\frac{\lambda^2}{8}\right)^2 = 1 + \frac{\lambda^2}{4} + \frac{\lambda^4}{64}, \tag{22}$$

$$A_7(\lambda) \equiv 1 + \frac{\lambda^2/4}{1-\lambda^2/16} = 1 + \frac{\lambda^2}{4} + \sum_{n=2}^{\infty} \frac{1}{2^{4n-2}}\lambda^{2n}, \tag{23}$$

$$A_8(\lambda) \equiv 3 - \sqrt{4-\lambda^2} = 1 + \frac{\lambda^2}{4} - \sum_{n=2}^{\infty} \frac{(-1/2)_n}{n!2^{2n-1}}\lambda^{2n}, \tag{24}$$

$$A_{10}(\lambda) \equiv \frac{64-3\lambda^4}{64-16\lambda^2} = 1 + \frac{\lambda^2}{4} + \sum_{n=2}^{\infty} \frac{1}{2^{2n+2}}\lambda^{2n}, \tag{25}$$

$$A_{11}(\lambda) \equiv \frac{3}{2} + \frac{\lambda^2}{8} - \frac{1}{2}\left(1-\frac{\lambda^2}{2}\right)^{1/2} = 1 + \frac{\lambda^2}{4} - \sum_{n=2}^{\infty} \frac{(-1/2)_n}{n!2^{n+1}}\lambda^{2n}, \tag{26}$$

$$A_{13}(\lambda) \equiv 1 + \frac{3\lambda^2}{10+\sqrt{4-3\lambda^2}} = 1 + \frac{\lambda^2}{4} + \sum_{n=2}^{\infty} \beta_n^{(13)}\lambda^{2n}, \tag{27}$$

$$A_4(\lambda) \equiv \frac{3}{2} - \frac{1}{2}(1-\lambda^2)^{1/2} = 1 + \frac{\lambda^2}{4} - \frac{1}{2}\sum_{n=2}^{\infty} \frac{(-1/2)_n}{n!}\lambda^{2n}, \tag{28}$$

$$A_3(\lambda) \equiv \frac{2}{1+\sqrt{1-\lambda^2}} = 1 + \frac{\lambda^2}{4} + \sum_{n=2}^{\infty} \beta_n^{(3)}\lambda^{2n}. \tag{29}$$

Proof of Proposition 2.3. We seek to establish the following inequalities regarding the specified Maclaurin series coefficients:

$$\beta_n^{(1)} \leq \beta_n^{(6)} \leq \beta_n^{(7)} \leq \beta_n^{(8)} \leq \beta_n^{(10)} \leq \beta_n^{(11)} \leq \beta_n^{(13)} \leq \alpha_n \leq \beta_n^{(4)} \leq \beta_n^{(3)} \tag{30}$$

for all $n \geq 0$. Referring to (21)–(29), we note that the inequalities in (30) are trivial for $n = 0$ and $n = 1$. Thus we must verify (30) for all $n \geq 2$. The first two inequalities are immediate while the next three inequalities follow directly by induction. We now proceed to prove the remaining inequalities in (30).

• Claim I. $\beta_n^{(11)} \le \beta_n^{(13)} \le \alpha_n$ *for all* $n \ge 2$.

Let $\beta_n \equiv \beta_n^{(13)}$ and $\gamma_n \equiv \beta_n^{(11)}$, where $\beta_n^{(11)} \equiv \frac{-(-1/2)_n}{n!2^{n+1}}$ for $n \ge 2$ (see (26)) and recall that $\alpha_n \equiv (\frac{(-1/2)_n}{n!})^2$. The nonzero Maclaurin series coefficients of Ramanujan's second estimate A_{13} can be shown to satisfy (see the appendix) $\beta_0 = 1$, $\beta_1 = 1/4$, $\beta_2 = 1/64$, and

$$\beta_n = \phi_{n-1} - 2^{-5}\beta_{n-1} \quad \text{for all } n \ge 3, \quad \text{where} \quad \phi_n \equiv -\frac{(-1/2)_n(3/4)^n}{16 \cdot n!}. \tag{31}$$

Applying (31) twice, we have

$$\beta_n = \phi_{n-1} - 2^{-5}\phi_{n-2} + 2^{-10}\beta_{n-2} \quad \text{for all } n \ge 4. \tag{32}$$

Direct calculation reveals that Claim I holds for $n = 2, 3, 4$. That is, $\gamma_n \le \beta_n \le \alpha_n$ for $n = 2, 3, 4$. Now let $n \ge 5$ and suppose that

$$\gamma_k \le \beta_k \le \alpha_k \quad \text{for all } k = 2, \dots, n-1. \tag{33}$$

Then (32) and (33) together imply that

$$\begin{aligned} &\phi_{n-1} - 2^{-5}\phi_{n-2} + 2^{-10}\gamma_{n-2} \\ &\le \overbrace{\phi_{n-1} - 2^{-5}\phi_{n-2} + 2^{-10}\beta_{n-2}}^{\beta_n} \le \phi_{n-1} - 2^{-5}\phi_{n-2} + 2^{-10}\alpha_{n-2}. \end{aligned} \tag{34}$$

It can be shown (see the appendix) that

$$\gamma_n \le \phi_{n-1} - 2^{-5}\phi_{n-2} + 2^{-10}\gamma_{n-2} \quad \text{and} \tag{35}$$

$$\alpha_n \ge \phi_{n-1} - 2^{-5}\phi_{n-2} + 2^{-10}\alpha_{n-2} \tag{36}$$

for all $n \ge 5$. Therefore, using inequalities (34)–(36) and induction, we have $\gamma_n \le \beta_n \le \alpha_n$ for all $n \ge 2$. This completes the proof of Claim I.

• Claim II. $\alpha_n \le \beta_n^{(4)} \le \beta_n^{(3)}$ *for all* $n \ge 2$.

If we now apply (28), the first inequality in Claim II becomes

$$\alpha_n \equiv \left(\frac{(-1/2)_n}{n!}\right)^2 \le \frac{-(-1/2)_n}{2 \cdot n!} \equiv \beta_n^{(4)} \quad \text{for all } n \ge 2.$$

This is equivalent to

$$\frac{-2(-1/2)_n}{n!} \le 1 \quad \text{for all } n \ge 2$$

which follows by induction. The second inequality in Claim II involves the Maclaurin series coefficients of Sipos and Ekwall's approximation A_3 which can be shown to satisfy the following recursive relationship (see the appendix): $\beta_0^{(3)} = 1$, $\beta_1^{(3)} = 1/4$, $\beta_2^{(3)} = 1/8$, and

$$\beta_n^{(3)} = -\frac{1}{2}\sum_{k=0}^{n-1} \frac{(-1/2)_{n-k}}{(n-k)!}\beta_k^{(3)} \quad \text{for all } n \ge 2. \tag{37}$$

Note that

$$-\frac{1}{2}\sum_{k=0}^{n-1}\frac{(-1/2)_{n-k}}{(n-k)!}\beta_k^{(3)}=\frac{-(-1/2)_n}{2\cdot n!}-\frac{1}{2}\sum_{k=1}^{n-1}\frac{(-1/2)_{n-k}}{(n-k)!}\beta_k^{(3)} \tag{38}$$

for all $n\geq 2$, and

$$\frac{-(-1/2)_{n-k}}{2\cdot(n-k)!}\beta_k^{(3)}>0 \quad \text{for } k=1,\dots,n-1. \tag{39}$$

Therefore, (37)–(39) together yield

$$\beta_n^{(4)}\equiv\frac{-(-1/2)_n}{2\cdot n!}\leq\beta_n^{(3)} \quad \text{for all } n\geq 2.$$

This concludes the proof of Claim II and Proposition 2.3. □

Remarks on the Proof of Proposition 2.4. From (8), we have that $c_n\equiv\dfrac{(1/2)^n(-1/2)_n}{n!}$ for all $n\geq 1$. By induction, it can be shown that

$$\frac{(1/2)_n(-1/2)_n}{n!\cdot n!}\leq c_n \quad \text{for all } n\geq 1.$$

In an earlier paper (see [9]), the authors use the logarithmic derivative and Cauchy products to obtain the recursive relationship for b_n (with b_n as defined in (7)) given by

$$b_{n+1}=\frac{1}{2(n+1)}\left[\left(\frac{5}{4}n-\frac{1}{2}\right)b_n-\sum_{k=0}^{n-2}(k+1)b_{k+1}\frac{\left(-\frac{1}{4}\right)_{n-k}}{(n-k)!}\right]. \tag{40}$$

Theorem 2.1, together with (40), was then used (see [9]) to establish that

$$b_n\leq\frac{(1/2)_n(-1/2)_n}{n!\cdot n!} \quad \text{for all } n\geq 1. \qquad \square$$

4. Proofs of general results involving $_3F_2$. We will make use of the following classical identities which we include for the reader's convenience ($F\equiv {}_3F_2$).

IDENTITY 1 {see [13, p. 440, eq. (33)]}.

$$\begin{aligned}&F(\rho,a,b;c,\sigma;1)-F(\rho+1,a,b;c,\sigma+1;1)\\&\qquad=\frac{-ab(\sigma-\rho)}{c\sigma(\sigma+1)}\cdot F(\rho+1,a+1,b+1;c+1,\sigma+2;1).\end{aligned}$$

IDENTITY 2 {see [11, p. 59, eq. (3.1.1)]}.

$$F(-n,a,b;c,d;1)=\frac{(d-b)_n}{(d)_n}\cdot F(-n,c-a,b;c,1+b-d-n;1).$$

IDENTITY 3 {see [13, p. 440, eq. (26)]}.

$$\sigma\cdot F(\rho,a,b;c,\sigma;1)=\rho\cdot F(\rho+1,a,b;c,\sigma+1;1)+(\sigma-\rho)\cdot F(\rho,a,b;c,\sigma+1;1).$$

IDENTITY 4 {see [14, p. 82, eq. (14)]}.

$$(a_1-a_2)\cdot F(a_1,a_2,a_3;b_1,b_2;z)=a_1\cdot F(a_1+1,a_2,a_3;b_1,b_2;z)-a_2\cdot F(a_1,a_2+1,a_3;b_1,b_2;z).$$

Identity 5 {see [13, p. 440, eq. (30)]}.

$$F\left(\sigma,a,b;c,d;1\right)-F\left(\sigma+1,a,b;c,d;1\right)=\frac{-ab}{cd}\cdot F\left(\sigma+1,a+1,b+1;c+1,d+1;1\right).$$

Proof of Theorem 2.5. Define $T_n\equiv F\left(-n,a,b;a+b+2,\varepsilon-n+1;1\right)$, where $F\equiv {}_3F_2$. Let $1>a\geq b>-1$ and $1>\varepsilon\geq\frac{(a+1)(b+2)}{a+b+4}$. For $n\geq 2$, it follows that

$$\begin{aligned}
T_{n+1}-T_n &= F\left(-n-1,a,b;a+b+2,\varepsilon-n;1\right)-F\left(-n,a,b;a+b+2,\varepsilon-n+1;1\right)\\
&= \frac{-ab(\varepsilon+1)}{(\varepsilon-n)(\varepsilon-n+1)(a+b+2)}F\left(-n,a+1,b+1;a+b+3,\varepsilon-n+2;1\right)\\
&\quad \{\text{using Identity 1 with } \rho=-n-1,\ \sigma=\varepsilon-n\}\\
&= \frac{-ab(\varepsilon+1)}{(n-\varepsilon)(n-\varepsilon-1)(a+b+2)}\frac{(\varepsilon-n-b+1)_n}{(\varepsilon-n+2)_n}\\
&\quad \times F\left(-n,b+2,b+1;a+b+3,b-\varepsilon;1\right)\\
&\quad \{\text{using Identity 2}\}\\
&= \frac{-ab(\varepsilon+1)(b-\varepsilon)_n}{(n-\varepsilon)(n-\varepsilon-1)(a+b+2)(-1-\varepsilon)_n(b-\varepsilon)}\\
&\quad \times[(b+1)F\left(-n,b+2,b+2;a+b+3,b+1-\varepsilon;1\right)\\
&\quad +(-\varepsilon-1)F\left(-n,b+2,b+1;a+b+3,b+1-\varepsilon;1\right)],
\end{aligned}\tag{41}$$

where (41) follows from Identity 3 (with $\rho=b+1$, $\sigma=b-\varepsilon$) and the identity $(1-\alpha-n)_n=(-1)^n(\alpha)_n$.
Identity 4 (with $a_1=-n$ and $a_2=b+1$) implies that

$$\begin{aligned}
&F(-n,b+2,b+2;a+b+3,b+1-\varepsilon;1)\\
&\quad=\frac{1}{b+1}[(n+b+1)F\left(-n,b+2,b+1;a+b+3,b+1-\varepsilon;1\right)\\
&\quad+(-n)F\left(-n+1,b+2,b+1;a+b+3,b+1-\varepsilon;1\right)].
\end{aligned}\tag{42}$$

Now let $G_n=F\left(-n,b+2,b+1;a+b+3,b+1-\varepsilon;1\right)$ and use (41) and (42). Then we have that

$$\begin{aligned}
T_{n+1}-T_n &= \frac{-ab(\varepsilon+1)(b-\varepsilon)_n}{(n-\varepsilon)(n-\varepsilon-1)(a+b+2)(-1-\varepsilon)_n(b-\varepsilon)}\\
&\quad\times[(n+b+1)G_n-nG_{n-1}+(-\varepsilon-1)G_n]\\
&= \frac{-ab(\varepsilon+1)(b-\varepsilon)_n}{(n-\varepsilon)(n-\varepsilon-1)(a+b+2)(-1-\varepsilon)_n(b-\varepsilon)}\\
&\quad\times[n(G_n-G_{n-1})+(b-\varepsilon)G_n].
\end{aligned}\tag{43}$$

Applications of Identity 5 (with $\sigma=-n$) followed by Identity 2 yield

$$\begin{aligned}
G_n-G_{n-1} &= F\left(-n,b+2,b+1;a+b+3,b+1-\varepsilon;1\right)\\
&\quad -F\left(-n+1,b+2,b+1;a+b+3,b+1-\varepsilon;1\right)\\
&= \frac{-(b+2)(b+1)}{(a+b+3)(b+1-\varepsilon)}F\left(-n+1,b+3,b+2;a+b+4,b+2-\varepsilon;1\right)\\
&= \frac{-(b+2)(b+1)}{(a+b+3)(b+1-\varepsilon)}\cdot\frac{(-\varepsilon)_{n-1}}{(b+2-\varepsilon)_{n-1}}\\
&\quad\times F\left(-n+1,a+1,b+2;a+b+4,\varepsilon-n+2;1\right).
\end{aligned}\tag{44}$$

Identity 2 also implies that

$$G_n = \frac{(-\varepsilon)_n}{(b+1-\varepsilon)_n} F\left(-n, a+1, b+1; a+b+3, \varepsilon-n+1; 1\right). \tag{45}$$

Combining (43)–(45), we have

$$\begin{aligned} T_{n+1} - T_n = &\frac{-ab(\varepsilon+1)(b-\varepsilon)_n}{(n-\varepsilon)(n-\varepsilon-1)(a+b+2)(-1-\varepsilon)_n(b-\varepsilon)} \\ \times &\left[\frac{-n(b+2)(b+1)(-\varepsilon)_{n-1}}{(a+b+3)(b+1-\varepsilon)(b+2-\varepsilon)_{n-1}} F\left(-n+1, a+1, b+2; a+b+4, \varepsilon-n+2; 1\right)\right. \\ &\left. +(b-\varepsilon)\frac{(-\varepsilon)_n}{(b+1-\varepsilon)_n} F\left(-n, a+1, b+1; a+b+3, \varepsilon-n+1; 1\right)\right]. \end{aligned} \tag{46}$$

Now make use of $\frac{(b-\varepsilon)_n}{(b+1-\varepsilon)_n} = \frac{(b-\varepsilon)}{(n+b-\varepsilon)}$, $\frac{(-\varepsilon)_n}{(-1-\varepsilon)_n} = \frac{(-1-\varepsilon+n)}{(-1-\varepsilon)}$, $\frac{(-\varepsilon)_{n-1}}{(-1-\varepsilon)_n} = \frac{1}{(-1-\varepsilon)}$, and multiply both sides by $-ab$. Then (46) becomes

$$\begin{aligned} ab(T_n - T_{n+1}) = &\frac{(ab)^2(\varepsilon+1)}{(n-\varepsilon)(n-\varepsilon-1)(a+b+2)(b-\varepsilon)} \\ \times &\left[\frac{-n(b+2)(b+1)(b-\varepsilon)}{(a+b+3)(-1-\varepsilon)(n+b-\varepsilon)} F\left(-n+1, a+1, b+2; a+b+4, \varepsilon-n+2; 1\right)\right. \\ &\left. +(b-\varepsilon)\frac{(n-1-\varepsilon)(b-\varepsilon)}{(-1-\varepsilon)(n+b-\varepsilon)} F\left(-n, a+1, b+1; a+b+3, \varepsilon-n+1; 1\right)\right] \\ = &\frac{(ab)^2}{(n-\varepsilon)(n-\varepsilon-1)(a+b+2)(n+b-\varepsilon)} \\ &\times \left[\frac{n(b+2)(b+1)}{(a+b+3)} F\left(-(n-1), a+1, b+2; a+b+4, \varepsilon-(n-1)+1; 1\right)\right. \\ &\left. + (\varepsilon-b)(n-\varepsilon-1) F\left(-n, a+1, b+1; a+b+3, \varepsilon-n+1; 1\right)\right], \end{aligned} \tag{47}$$

where $n+b-\varepsilon > n-\varepsilon-1 > n-2 \geq 0$, $n-\varepsilon > 0$, and $\varepsilon - b > \varepsilon - \frac{(a+1)(b+2)}{a+b+4} \geq 0$. Since $1 > \varepsilon \geq \frac{(a+1)(b+2)}{a+b+4} > \frac{(a+1)(b+1)}{a+b+3}$, Theorem 2.1 implies that

$$F\left(-(n-1), a+1, b+2; a+b+4, \varepsilon-(n-1)+1; 1\right) \geq 0 \quad \text{and}$$
$$F\left(-n, a+1, b+1; a+b+3, \varepsilon-n+1; 1\right) \geq 0.$$

Therefore, (47) is the product and sum of nonnegative quantities and thus

$$ab(T_n - T_{n+1}) \geq 0 \quad \text{for all integers } n \geq 2. \qquad \square$$

In order to prove Corollary 2.6, we will make use of the following two lemmas.

LEMMA 4.1. *Let n be a positive integer and $0 < \varepsilon < 1$. Then*

$$\frac{(-n)_k}{(\varepsilon-n+1)_k} \geq 1 \quad \textit{for all } k = 0, \dots, n-1.$$

Proof of Lemma 4.1. Note that the desired inequality holds at $k = 0$. Now let $n \geq 2$ and suppose that

$$\frac{(-n)_k}{(\varepsilon-n+1)_k} \geq 1 \quad \text{for some } k \text{ with } 0 \leq k \leq n-2.$$

Then

$$\frac{(-n)_{k+1}}{(\varepsilon-n+1)_{k+1}} = \frac{(-n)_k(-n+k)}{(\varepsilon-n+1)_k(\varepsilon-n+1+k)} \geq \frac{(-n)_k}{(\varepsilon-n+1)_k} \geq 1. \qquad \square$$

LEMMA 4.2. *Define* $\psi_n(a,b,c,\varepsilon) \equiv \frac{(a)_n(b)_n(-n)_n}{n!(c)_n(\varepsilon-n+1)_n}$. *Let* (a,b,c,ε) *be in the domain of* ψ_n *for all* $n \geq 2$ *with* $\varepsilon < c-a-b$. *Then*

$$\lim_{n\to\infty} \psi_n(a,b,c,\varepsilon) = 0.$$

Proof of Lemma 4.2. Since $(1-c-n)_n = (-1)^n(c)_n$, it follows that

$$\psi_n = \frac{(a)_n(b)_n(1)_n}{n!(c)_n(-\varepsilon)_n} = \frac{\Gamma(a+n)\Gamma(b+n)\Gamma(c)\Gamma(-\varepsilon)}{\Gamma(a)\Gamma(b)\Gamma(c+n)\Gamma(-\varepsilon+n)} n^{c-a-b-\varepsilon} n^{a+b+\varepsilon-c}.$$

It is known that (see [1, p. 257, eq. (6.1.46)])

$$\lim_{n\to\infty} \frac{\Gamma(r+n)}{\Gamma(s+n)} n^{s-r} = 1.$$

If $a+b+\varepsilon-c<0$, then

$$\lim_{n\to\infty}\psi_n = \lim_{n\to\infty} \frac{\Gamma(a+n)\Gamma(b+n)\Gamma(c)\Gamma(-\varepsilon)}{\Gamma(a)\Gamma(b)\Gamma(c+n)\Gamma(-\varepsilon+n)} n^{c-a-b-\varepsilon} \cdot \lim_{n\to\infty} n^{a+b+\varepsilon-c} = 0. \qquad \square$$

Proof of Corollary 2.6. Let $1 > a \geq b > -1$ and $1 > \varepsilon \geq \frac{(a+1)(b+2)}{a+b+4}$ and define

$$T_n \equiv {}_3F_2\left(-n, a, b; a+b+2, \varepsilon-n+1; 1\right).$$

Theorem 2.5 implies that the sequence $\{abT_n\}_{n=2}^{\infty}$ is a monotone (nonincreasing) sequence. Now define

$$S_n \equiv 1 + \frac{(a)_n(b)_n(-n)_n}{n!(a+b+2)_n(\varepsilon-n+1)_n} + \sum_{k=1}^{n-1} \frac{(a)_k(b)_k}{k!(a+b+2)_k}.$$

Using the definition of $_3F_2$, Lemma 4.1, and the fact that $\dfrac{ab(a)_k(b)_k}{k!(a+b+2)_k} \geq 0$ for $k=1,\ldots,n-1$, we obtain

$$abT_n = ab + \frac{ab(a)_n(b)_n(-n)_n}{n!(a+b+2)_n(\varepsilon-n+1)_n} + \sum_{k=1}^{n-1} \frac{ab(a)_k(b)_k(-n)_k}{k!(a+b+2)_k(\varepsilon-n+1)_k} \geq abS_n$$

for all $n \geq 2$.
Applying Lemma 4.2 with $c = a+b+2$, we have

$$\lim_{n\to\infty} S_n = {}_2F_1\left(a, b; a+b+2; 1\right).$$

Since $abT_n \geq abS_n$ for all $n \geq 2$, it follows that $\{abT_n\}_{n=2}^{\infty}$ is a bounded monotone sequence. Thus

$$abT_n \geq \lim_{n\to\infty} abT_n \geq \lim_{n\to\infty} abS_n = ab \cdot {}_2F_1\left(a, b; a+b+2; 1\right) \quad \text{for all } n \geq 2. \qquad \square$$

Proof of Corollary 2.7. Choose $a = b = -1/2$ and $1 > \varepsilon \geq 1/4$ and define

$$T_n \equiv {}_3F_2\left(-n, -\frac{1}{2}, -\frac{1}{2}; 1, \varepsilon - n + 1; 1\right).$$

It is known that (see [14, p. 49])

$${}_2F_1\left(-\frac{1}{2}, -\frac{1}{2}; 1; 1\right) = \frac{4}{\pi}.$$

Corollary 2.6 implies that

$$T_n \geq T_{n+1} \geq \frac{4}{\pi} \quad \text{for all } n \geq 2. \qquad \square$$

5. Appendix.

5.1. Recursive relationship for Maclaurin series coefficients of Ramanujan's second estimate A_{13}. Writing $\beta_n \equiv \beta_n^{(13)}$, we have

$$3\lambda^2(10 - \sqrt{4 - 3\lambda^2}) = (A_{13}(\lambda) - 1)(10^2 - (\sqrt{4 - 3\lambda^2})^2) = (96 + 3\lambda^2) \sum_{n=1}^{\infty} \beta_n \lambda^{2n}$$

which implies that

$$10 - 2\left(1 - \frac{3}{4}\lambda^2\right)^{1/2} = (32 + \lambda^2) \sum_{n=1}^{\infty} \beta_n \lambda^{2n-2}.$$

Applying $(1 - x)^q = \sum_{n=0}^{\infty} \frac{(-q)_n}{n!} x^n$ and simplifying yields

$$8 - 2\sum_{n=1}^{\infty} \frac{(-1/2)_n (3/4)^n}{n!} \lambda^{2n} = 32\beta_1 + \sum_{n=1}^{\infty} (32\beta_{n+1} + \beta n)\lambda^{2n}.$$

Thus $\beta_0 = 1$, $\beta_1 = 1/4$, $\beta_2 = 1/64$, and

$$\beta_{n+1} = \frac{-(-1/2)_n (3/4)^n}{16 \cdot n!} - \frac{\beta_n}{32} \quad \text{for all } n \geq 1.$$

Letting $\phi_n \equiv -\dfrac{(-1/2)_n (3/4)^n}{16 \cdot n!}$, we obtain

$$\beta_{n+1} = \phi_n - 2^{-5}\beta_n \quad \text{for all } n \geq 2. \qquad \square$$

5.2. Recursive relationship for Maclaurin series coefficients of Sipos and Ekwall's estimate A_3. Writing $\beta_n \equiv \beta_n^{(3)}$ and using the Cauchy product, we have

$$2 = A_3(\lambda)(1 + \sqrt{1 - \lambda^2}) = \sum_{n=0}^{\infty} \beta_n \lambda^{2n} + \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(-1/2)_{n-k}}{(n-k)!} \beta_k \lambda^{2n}.$$

Thus $\beta_0^{(3)} = 1$, $\beta_1^{(3)} = 1/4$, $\beta_2^{(3)} = 1/8$, and

$$\beta_n^{(3)} = \frac{-1}{2} \sum_{k=0}^{n-1} \frac{(-1/2)_{n-k}}{(n-k)!} \beta_k^{(3)} \quad \text{for all } n \geq 2. \qquad \square$$

5.3. Establishing inequality (35). Let $\phi_n \equiv \frac{-(-1/2)_n(3/4)^n}{16\cdot n!}$, $\gamma_n \equiv \beta_n^{(11)} = \frac{-(-1/2)_n}{n!2^{n+1}}$, and $n \geq 4$. Inequality (35) claims that $\gamma_n \leq \phi_{n-1} - 2^{-5}\phi_{n-2} + 2^{-10}\gamma_{n-2}$.

Direct calculation reveals that the desired inequality holds for $n = 4, \ldots, 7$. Now suppose that $n \geq 7$. Since $\gamma_{n-2} > 0$, we have

$$\begin{aligned}
&\gamma_n - \phi_{n-1} + 2^{-5}\phi_{n-2} - 2^{-10}\gamma_{n-2} < \gamma_n - \phi_{n-1} + 2^{-5}\phi_{n-2} \\
&\quad = \frac{-(-1/2)_n}{n!2^{n+1}} + \frac{(-1/2)_{n-1}(3/4)^{n-1}}{16\cdot(n-1)!} - 2^{-9}\frac{(-1/2)_{n-2}(3/4)^{n-2}}{(n-2)!} \\
&\quad = -2^{-9}\frac{(-1/2)_{n-2}(3/4)^{n-2}}{(n-2)!}\cdot\left[\frac{2^9(4/3)^{n-2}(n-3/2)(n-5/2)}{n(n-1)2^{n+1}} - \frac{2^5(3/4)(n-5/2)}{(n-1)} + 1\right] \\
&\quad = -2^{-9}\frac{(-1/2)_{n-2}(3/4)^{n-2}}{(n-2)!}\cdot\left[\frac{2^{n+4}(n-3/2)(n-5/2)}{3^{n-2}n(n-1)} - \frac{24(n-5/2)}{(n-1)} + 1\right].
\end{aligned}$$

Since $\frac{(n-5/2)}{(n-1)} \geq \frac{1}{2}$, it follows that

$$\frac{2^{n+4}(n-3/2)(n-5/2)}{3^{n-2}n(n-1)} - \frac{24(n-5/2)}{(n-1)} + 1 \leq \frac{2^{n+4}(n-3/2)(n-5/2)}{3^{n-2}n(n-1)} - 11.$$

Thus

$$\begin{aligned}
&\gamma_n - \phi_{n-1} + 2^{-5}\phi_{n-2} - 2^{-10}\gamma_{n-2} \\
&\quad < -2^{-5}\cdot 9\frac{(-1/2)_{n-2}(3/4)^{n-2}}{(n-2)!}\cdot\left[\frac{2^n(n-3/2)(n-5/2)}{3^n n(n-1)} - \frac{11\cdot 2^{-4}}{3^2}\right] \\
&\quad \leq \frac{-9(-1/2)_{n-2}(3/4)^{n-2}}{32(n-2)!}\cdot\left[\left(\frac{2}{3}\right)^n - \frac{11}{144}\right] \\
&\quad \leq \frac{-9(-1/2)_{n-2}(3/4)^{n-2}}{32(n-2)!}\cdot\left[\left(\frac{2}{3}\right)^7 - \frac{11}{144}\right] < 0.
\end{aligned}$$

Hence the claim in (35) is established. $\square$

5.4. Establishing inequality (36). Let $\phi_n \equiv -\frac{(-1/2)_n(3/4)^n}{16\cdot n!}$, $\alpha_n \equiv (\frac{(-1/2)_n}{n!})^2$, and $n \geq 4$.

Inequality (36) claims that $\phi_n - 2^{-5}\phi_{n-1} + 2^{-10}\alpha_{n-1} \leq \alpha_{n+1}$. Note that

$$\begin{aligned}
&\phi_n - 2^{-5}\phi_{n-1} + 2^{-10}\alpha_{n-1} - \alpha_{n+1} \\
&= -\frac{(-1/2)_n(3/4)^n}{16\cdot n!} + 2^{-5}\frac{(-1/2)_{n-1}(3/4)^{n-1}}{16\cdot(n-1)!} + 2^{-10}\left(\frac{(-1/2)_{n-1}}{(n-1)!}\right)^2 - \left(\frac{(-1/2)_{n+1}}{(n+1)!}\right)^2 \\
&= \frac{(-1/2)_{n-1}}{(n-1)!}\left\{\frac{-(n-3/2)3^n}{n2^{2n+4}} + \frac{3^{n-1}}{2^{2n+7}} + \frac{(-1/2)_{n-1}}{2^{10}(n-1)!} - \frac{(n-3/2)(n-1/2)(-1/2)_{n+1}}{n(n+1)\cdot(n+1)!}\right\} \\
&= \frac{(-1/2)_{n-1}}{(n-1)!}\left\{\frac{3^{n-1}}{2^{2n+4}}\left[\frac{3(3/2-n)}{n} + \frac{1}{2^3}\right] + \frac{(-1/2)_{n-1}}{(n-1)!}\left[\frac{1}{2^{10}} - \frac{(n-3/2)^2(n-1/2)^2}{n^2(n+1)^2}\right]\right\} \\
&= \frac{(-1/2)_{n-1}}{(n-1)!}\left\{\frac{U(n)}{V(n)} + 1\right\}V(n),
\end{aligned}$$

where

$$U(n) \equiv \frac{3^{n-1}}{2^{2n+4}}\left[\frac{3(3/2-n)}{n} + \frac{1}{2^3}\right]$$

and

$$V(n) \equiv \frac{(-1/2)_{n-1}}{(n-1)!}\left[\frac{1}{2^{10}} - \frac{(n-3/2)^2(n-1/2)^2}{n^2(n+1)^2}\right].$$

It follows that $V(n) > 0$. Now let $W(n) \equiv U(n)/V(n)$. Since $(-1/2)_{n-1} < 0$, we will be finished if we can show that $W(n) + 1 > 0$ for all $n \geq 4$. Direct calculation again yields $W(4) + 1 > 0$. For $n \geq 4$, it is easy to check that

$$\begin{aligned} W(n+1) - W(n) &= \frac{\frac{3^n}{2^{2n+6}}\left[\frac{3(1/2-n)}{n+1} + \frac{1}{2^3}\right]}{\frac{(-1/2)_n}{n!}\left[\frac{1}{2^{10}} - \frac{(n-1/2)^2(n+1/2)^2}{(n+1)^2(n+2)^2}\right]} \\ &\quad - \frac{\frac{3^{n-1}}{2^{2n+4}}\left[\frac{3(3/2-n)}{n} + \frac{1}{2^3}\right]}{\frac{(-1/2)_{n-1}}{(n-1)!}\left[\frac{1}{2^{10}} - \frac{(n-3/2)^2(n-1/2)^2}{n^2(n+1)^2}\right]} \\ &= \left\{\frac{\frac{3^{n-1}}{2^{2n+4}}}{\frac{(-1/2)_{n-1}}{(n-1)!}}\right\}\left\{\frac{3n}{4(n-3/2)}Z(n+1) - Z(n)\right\}, \end{aligned} \tag{48}$$

where $Z(n) \equiv [\frac{3(3/2-n)}{n} + \frac{1}{2^3}]/[\frac{1}{2^{10}} - \frac{(n-3/2)^2(n-1/2)^2}{n^2(n+1)^2}]$. Direct calculation reveals that the expression in (48) is nonnegative for $n = 4$ and $n = 5$. For $n \geq 6$, it can be shown by a straightforward calculation that $0 < Z(n+1) \leq Z(n)$. Hence $\frac{3n}{4(n-3/2)}Z(n+1) - Z(n) \leq Z(n+1) - Z(n) \leq 0$ for all $n \geq 6$. Thus $W(n+1) - W(n) \geq 0$ for all $n \geq 4$ since $(-1/2)_{n-1} < 0$. Therefore, $W(n) + 1 \geq W(4) + 1 > 0$ for all $n \geq 4$. This establishes the claim in (36). $\square$

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