Three Extremal Problems for Hyperbolically Convex Functions

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Dedicated to the memory of Walter Hengartner.

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Abstract. In this paper we apply a variational method to three extremal problems for hyperbolically convex functions posed by Ma and Minda and Pommerenke [7, 16]. We first consider the problem of extremizing $\operatorname{Re}\frac{f(z)}{z}$. We determine the minimal value and give a new proof of the maximal value previously determined by Ma and Minda. We also describe the geometry of the hyperbolically convex functions $f(z) = \alpha z + a_2 z^2 + a_3 z^3 + \ldots$ which maximize $\operatorname{Re} a_3$.

1. Introduction

A classical problem in Geometric Function Theory is to maximize the value of a given functional over a given class of analytic functions. Recent papers have extended this problem and its study to functionals on hyperbolically convex functions. In particular, these functions were studied by Beardon in [6], Ma and Minda in [7, 8], Solynin in [17, 18], Mejía and Pommerenke in [9, 10, 11, 12, 13] and Mejía, Pommerenke, and Vasil′ev in [14]. This flurry of activity has produced a number of open problems and conjectures.

Recently, in [2, 4] we developed a variational technique for hyperbolically convex functions based on Julia's variational formula and applied it to several of these problems and conjectures.

For $z \in \mathbb{C}$, let $\operatorname{Re}\{z\}$ = real part z and let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. With the metric $\lambda(z)|dz| = \frac{2|dz|}{1-|z|^2}$, $\mathbb{D}$ forms the Poincaré model of the hyperbolic plane. In this model, hyperbolic geodesics in $\mathbb{D}$ are subarcs of Euclidean circles which intersect $\partial\mathbb{D}$ orthogonally. A set $S \subset \mathbb{D}$ is *hyperbolically convex* if for any two points z_1 and z_2 in S, the hyperbolic geodesic connecting z_1 to z_2 lies entirely inside of S. Important examples of hyperbolically convex regions are the fundamental domains of Fuchsian groups.

We will say that a function $f : \mathbb{D} \to \mathbb{D}$ is *hyperbolically convex* if f is analytic and univalent on $\mathbb{D}$ and if $f(\mathbb{D})$ is hyperbolically convex. The set of all hyperbolically convex functions f which satisfy $f(0) = 0, f'(0) > 0$ will be denoted by H.

A *hyperbolic polygon* is a simply connected subset of $\mathbb{D}$, which contains the origin and which is bounded by a Jordan curve consisting of a finite collection of hyperbolic geodesics and arcs of the unit circle. The bounding geodesics will be referred to as *proper sides*. We will let H^{poly} denote the subset of H of all functions mapping $\mathbb{D}$ onto hyperbolic polygons. Further, we will let H^n denote the subclass of H^{poly} of all functions mapping $\mathbb{D}$ onto polygons with at most n

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proper sides. It can be shown that H^{poly} is dense in H. Moreover, $H \cup \{0\}$ and $H^n \cup \{0\}$, for each n, are compact.

Each function $f \in H$ satisfies Schwarz's Lemma and, hence, $f'(0) \leq 1$. For $0 < \alpha \leq 1$, let $H_\alpha = \{f \in H : f(z) = \alpha z + a_2 z^2 + a_3 z^3 + \cdots\}$. As an aside, we note that H_1 consists only of the identity map.

Ma and Minda [7, 8] and Mejía and Pommerenke [9] describe the geometry of the function

$$k_\alpha(z) \equiv \frac{2\alpha z}{(1-z) + \sqrt{(1-z)^2 + 4\alpha^2 z}}$$

which belongs to H_α and which maps $\mathbb{D}$ to a hyperbolic polygon bounded by exactly one proper side. As a consequence of the normalizations, it can easily be seen that $H_\alpha \cap H^1$ consists solely of k_α and its rotations.

Ma and Minda [7] and Pommerenke [16] posed the following three problems for hyperbolically convex functions, whose solutions did not follow from the techniques in [9, 7, 8, 11, 12, 13] .

Problem 1. Fix $0 < \alpha < 1$ and let $f \in H_\alpha$. For $z \in \mathbb{D} \setminus \{0\}$, find

$$\min_{f \in H_\alpha} \operatorname{Re} \frac{f(z)}{z}. \tag{1}$$

Problem 2. Fix $0 < \alpha < 1$ and let $f \in H_\alpha$ with $f(z) = \alpha z + a_2 z^2 + a_3 z^3 + \cdots$. Find

$$\max_{f \in H_\alpha} \operatorname{Re}\ a_3. \tag{2}$$

Problem 3. Let $f \in H$ with $f(z) = \alpha z + a_2 z^2 + a_3 z^3 + \cdots$. Find

$$\max_{f \in H} \operatorname{Re}\ a_3. \tag{3}$$

Applying the variational methods developed in [2, 4], we have the following resolutions for Problems 1, 2 and 3.

Theorem 1.1. *For any $z \in \mathbb{D} \setminus \{0\}$ and fixed $0 < \alpha < 1$, let $f \in H_\alpha$ be extremal for $L(f) = \operatorname{Re} \frac{f(z)}{z}$ over H_α. Then the extremal value (maximum or minimum) for L over H_α can be obtained from a hyperbolically convex function f which maps $\mathbb{D}$ onto a hyperbolic polygon with exactly one proper side. Specifically, for $z \in \mathbb{D} \setminus \{0\}$ and $0 < \alpha < 1$*

$$\max_{f \in H_\alpha} \operatorname{Re} \frac{f(z)}{z} = \frac{k_\alpha(r)}{r},\ r = |z|$$

and

$$\min_{f \in H_\alpha} \operatorname{Re} \frac{f(z)}{z} = \frac{k_\alpha(-r)}{-r},\ r = |z|.$$

Theorem 1.2. *Fix $0 < \alpha < 1$. Then the maximal value for $L(f) = \operatorname{Re} a_3$ over H_α is obtained by a hyperbolically convex function $f(z) = \alpha z + a_2 z^2 + a_3 z^3 + \cdots$ which maps $\mathbb{D}$ onto a hyperbolic polygon with at most two proper sides.*

Theorem 1.3. *The maximal value for $L(f) = \operatorname{Re} a_3$ over H is obtained by a hyperbolically convex function $f(z) = \alpha z + a_2 z^2 + a_3 z^3 + \cdots$ which maps $\mathbb{D}$ onto a hyperbolic polygon with at most two proper sides.*

Remark 1.1. *The maximum value of $\operatorname{Re} \frac{f(z)}{z}$ for hyperbolically convex functions was first given using different methods by Ma and Minda [7]. Minda and Ma also observed in [7] that k_α cannot be extremal for (2) when $\alpha = 1/2$. Hence, the reduction in Theorem 1.2 of the extremal function to a hyperbolically convex function with two proper sides is best possible.*

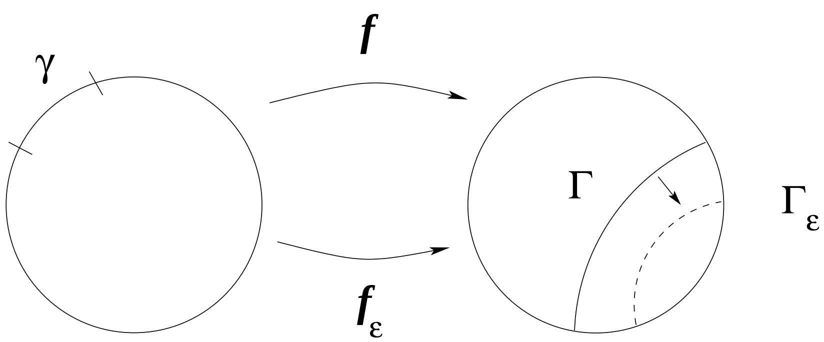

Figure 1. The variation produced by "pushing out" a side.

2. Variations for H^{poly}

As our primary tool for solving these three problems, we will modify an extension of the Julia variation as developed by the first author and J. Lewis [5, 3].

Let Ω be a region bounded by a piecewise analytic curve Γ and $\phi(w)$ be a positive piecewise C^1 function on Γ, vanishing where Γ is not analytic. Denote the outward pointing unit normal vector at each point w where Γ is smooth by $n(w)$. For ϵ near 0, construct a new curve

$$\Gamma_\epsilon = \{w + \epsilon\phi(w)n(w) : \ w \in \Gamma\}$$

and let Ω_ϵ be the new region bounded by Γ_ϵ.

Notice that if ϵ is sufficiently small and $\epsilon > 0$, then Γ is "pushed outside" the domain, while if $\epsilon < 0$, then Γ is "pushed inside" the domain. With the above notation, we state Julia's modification of the Hadamard variational formula; see [5] for details.

Lemma 2.1. *Let f be a conformal map from $\mathbb{D}$ onto Ω with $f(0) = 0$, and suppose f has a continuous extension to $\partial\mathbb{D}$, which we also denote by f. Then, a similarly normalized conformal map from $\mathbb{D}$ onto Ω_ϵ is given by*

$$f_\epsilon(z) = f(z) + \frac{\epsilon z f'(z)}{2\pi} \int_{\partial\mathbb{D}} \frac{1+\xi z}{1-\xi z}\, d\Psi + o(\epsilon), \tag{4}$$

where $d\Psi = \frac{\phi(f(\xi))}{|f'(\xi)|} d\theta$, $\xi = e^{i\theta}$, and the $o(\epsilon)$ term is continuously differentiable in ϵ for each $z \in \mathbb{D}$. Furthermore, the change in mapping radius between f_ϵ and f is given by

$$\Delta mr(f_\epsilon, f) = \frac{\epsilon f'(0)}{2\pi} \int_{\partial\mathbb{D}} d\Psi + o(\epsilon). \tag{5}$$

Notice the restriction that ϕ vanish at the points of non-smoothness of $\partial\Omega$ is a strong one. It implies, for example, that while we can vary the sides of a hyperbolic polygon, we cannot move any of the vertices. However, it follows from the work of the first author and J. Lewis [3] that such an extended version of the Julia variation is possible (except at internal cusps, ie., where two sides meet at an angle of measure 2π, which do not occur for hyperbolically convex polygons). Moreover, they showed that the resulting function will agree with the Julia variational formula up to $o(\epsilon)$ terms.

In [2, 4], we proved the following two lemmas which describe variations for functions in H^{poly} which preserve hyperbolic convexity. First, if f maps onto a hyperbolically convex polygon Ω with a side Γ, we can "push" Γ to a nearby geodesic Γ_ϵ in such a way that the varied function

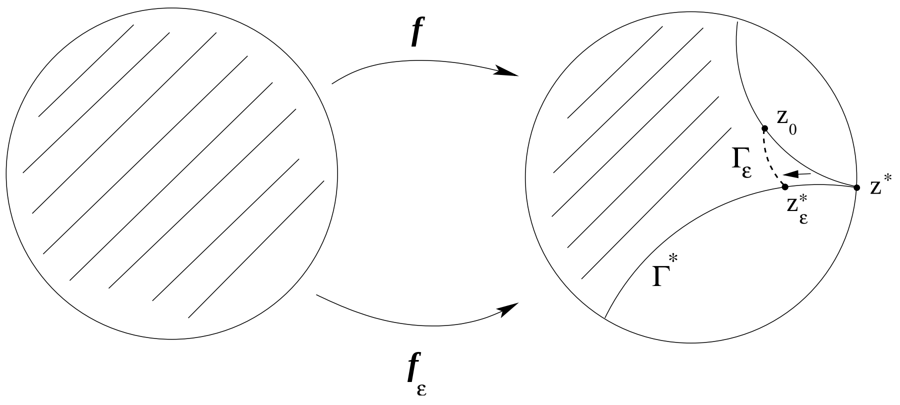

FIGURE 2. The variation produced by pushing in one end of a side.

f_ϵ will still be hyperbolically convex. Although the control ϕ can depend on ϵ, it was shown in [2] that the variation in ϕ can be absorbed into the $o(\epsilon)$ terms. See Figure 1.

Lemma 2.2. *Suppose $f \in H^n$ and f is not constant. If $\Gamma = f(\gamma)$ is a proper side of $\Omega = f(\mathbb{D})$, then for ϵ sufficiently small there exists a variation $f_\epsilon \in H^n$ which "pushes" Γ either in or out to a nearby geodesic Γ_ϵ, where $\Gamma_\epsilon \to \Gamma$ as $\epsilon \to 0$. Moreover, f_ϵ agrees with the Julia Variational Formula up to $o(\epsilon)$ terms.*

If Γ intersects another side Γ^* at z^*, and $z_0 \in \Gamma$, then we can vary f so as to replace the portion of Γ between z_0 and z^* with a new geodesic between z_0 and some $z^*_\epsilon \in \Gamma^*$. See Figure 2.

Lemma 2.3. *Suppose $f \in H^n$, f is not constant, and $\Gamma = f(\gamma)$ is a proper side of $f(\mathbb{D})$ meeting a side Γ^*. Then, there exists a variation $f_\epsilon \in H_{n+1}$ which adds a side to $f(\mathbb{D})$ by pushing one end of Γ to a nearby side Γ_ϵ. That is, $f_\epsilon(\mathbb{D})$ is a hyperbolic polygon whose sides are the same as those of f, except that one end of Γ has been replaced by Γ_ϵ and Γ^* has been shortened. Moreover, f_ϵ agrees with the Julia Variational Formula up to $o(\epsilon)$ terms.*

Remark 2.1. *Notice that in order to maintain hyperbolic convexity, we can only push in the "end" of a side, that is, a subarc that ends at a vertex of the polygon. However, we can choose this subarc to be as long, or more importantly, as short, as we choose.*

3. Proofs

3.1. Proof of Theorem 1.1. Choose a point $z \in \mathbb{D} \setminus \{0\}$. We will consider explicitly the problem of minimizing $\mathrm{Re}\left\{\frac{f(z)}{z}\right\}$ over H_α. The case for maximizing $\mathrm{Re}\left\{\frac{f(z)}{z}\right\}$ over H_α is analogous.

Let $H^n_\alpha = H_\alpha \cap H^n$. Suppose that $f \in H^n_\alpha$ is extremal for (1) over H^n_α for some $n \geq 3$ and that $f(\mathbb{D})$ has at least three proper sides, say $\Gamma_j,\ j = 1, 2, 3$. Let $\gamma_j = f^{-1}(\Gamma_j),\ j = 1, 2, 3$. For each side Γ_j, apply the variation described in Lemma 2.2 to Γ_j, with control $\epsilon_j = \epsilon\lambda_j$. Let f_ϵ be the varied function obtained by varying each of the three sides $\Gamma_j,\ j = 1, 2, 3$, of f. From Lemma 2.1, we have

$$\frac{f_\epsilon(z)}{z} = \frac{f(z)}{z} + \epsilon \sum_{j=1}^{3} \frac{\lambda_j}{2\pi} \int_{\gamma_j} f'(z) \frac{1 + \xi z}{1 - \xi z}\, d\Psi + o(\epsilon)$$

and

$$\Delta mr(f_\epsilon, f) = \epsilon \sum_{j=1}^{3} \frac{\lambda_j \alpha}{2\pi} \int_{\gamma_j} d\Psi + o(\epsilon)$$

Hence, we can write

$$L(f_\epsilon) = L(f) + \epsilon \operatorname{Re} \left\{ \sum_{j=1}^{3} \frac{\lambda_j}{2\pi} \int_{\gamma_j} f'(z) \frac{1+\xi z}{1-\xi z} \, d\Psi + o(\epsilon) \right\}.$$

If $\frac{\partial \Delta mr(f_\epsilon, f)}{\Delta \epsilon}\big|_{\epsilon=0} = 0$, then f_ϵ will also lie in H_α^n for ϵ sufficiently small. If in addition, $\frac{\partial L(f_\epsilon)}{\partial \epsilon}\big|_{\epsilon=0}$ is non-zero, then the value of $L(f_\epsilon)$ can be made smaller than the value of $L(f)$ for some ϵ near 0. Thus f cannot be extremal for (1) in H_α^n. Using the above representations for $L(f_\epsilon)$ and $\Delta mr(f_\epsilon, f)$ we obtain

$$\frac{\partial L(f_\epsilon)}{\partial \epsilon}\bigg|_{\epsilon=0} = \operatorname{Re} \left\{ \sum_{j=1}^{3} \frac{\lambda_j}{2\pi} \int_{\gamma_j} f'(z) \frac{1+\xi z}{1-\xi z} \, d\Psi \right\} \tag{6}$$

and

$$\frac{\partial \Delta mr(f_\epsilon, f)}{\partial \epsilon}\bigg|_{\epsilon=0} = \sum_{j=1}^{3} \frac{\lambda_j \alpha}{2\pi} \int_{\gamma_j} d\Psi. \tag{7}$$

Let $Q(\xi) = f'(z)\frac{1+\xi z}{1-\xi z}$. As $d\Psi$ is real valued, we can apply the mean value theorem for integrals and rewrite (6) as

$$\frac{\partial L(f_\epsilon)}{\partial \epsilon}\bigg|_{\epsilon=0} = \left\{ \sum_{j=1}^{3} \frac{\lambda_j}{2\pi} \operatorname{Re} Q(\xi_j) \int_{\gamma_j} d\Psi \right\} \tag{8}$$

where ξ_j belongs to the interior of the arc γ_j.

Since the kernel Q of our integral is bilinear in ξ, it maps $\partial\mathbb{D}$ to a circle Λ. Hence, for $|z| < 1$, not all three of the points $Q(\xi_j)$, $j = 1, 2, 3$, can have the same real part. Without loss of generality suppose that $\operatorname{Re} Q(\xi_1) > \operatorname{Re} Q(\xi_2)$. Then, we can push Γ_1 in, Γ_2 out (and not vary Γ_3) so as to decrease the value of $L(f_\epsilon)$ from the value of $L(f)$ within the class H_α^n. Specifically, choose $\lambda_1 < 0 < \lambda_2$ (and $\lambda_3 = 0$) so that $\frac{\partial \Delta mr(f_\epsilon, f)}{\epsilon}\big|_{\epsilon=0} = 0$ in (7). Then, we have from (8)

$$\begin{aligned} \frac{\partial L(f_\epsilon)}{\partial \epsilon}\bigg|_{\epsilon=0} &= \operatorname{Re} Q(\xi_1) \frac{\lambda_1}{2\pi} \int_{\gamma_1} d\Psi + \operatorname{Re} Q(\xi_2) \frac{\lambda_2}{2\pi} \int_{\gamma_2} d\Psi \\ &< \operatorname{Re} Q(\xi_1) \left(\frac{\lambda_1}{2\pi} \int_{\gamma_1} d\Psi + \frac{\lambda_2}{2\pi} \int_{\gamma_2} d\Psi \right) \\ &= 0. \end{aligned} \tag{9}$$

Thus, f is not extremal for L in H_α^n. Consequently, if f is extremal in H_α^n, $n \geq 3$, then $f \in H_\alpha^2 \subset H_\alpha^n$, that is, the extremal f can have at most two proper sides.

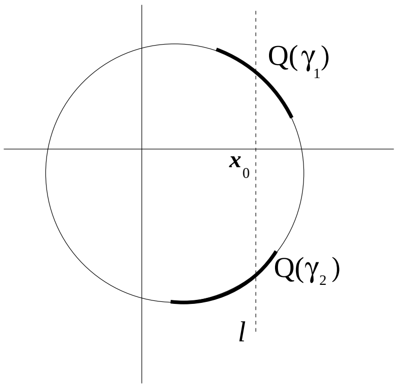

FIGURE 3. The endpoints of $Q(\gamma_j)$, $j = 1, 2$, must lie on opposite sides of the line l determined by $x = x_0 = \operatorname{Re} Q(\xi_1) = \operatorname{Re} Q(\xi_2)$.

We will now argue that f can actually have at most one side using an argument similar to the *Step Down Lemma* in [2]. Consider H_α^n, $n \geq 3$, and let f be extremal in H_α^n for (1). By the above argument, $f(\mathbb{D})$ can have at most two sides. Suppose $f(\mathbb{D})$ has exactly two proper sides, say $\Gamma_j,\ j = 1, 2$. As above, for each side Γ_j, apply the variation described in Lemma 2.2 to Γ_j, with control $\epsilon_j = \epsilon\lambda_j$. Let f_ϵ be the varied function obtained by varying each of the two sides $\Gamma_j,\ j = 1, 2$, of f. If in the formulation for $\frac{\partial L(f_\epsilon)}{\partial \epsilon}\Big|_{\epsilon=0}$ in (8), suitably modified to reflect only moving two sides, $\operatorname{Re} Q(\xi_1) \neq \operatorname{Re} Q(\xi_2)$, then the same argument used above eliminates f from being extremal. So we conclude that $\operatorname{Re} Q(\xi_1) = \operatorname{Re} Q(\xi_2) = x_0$. Thus, for each proper side $\Gamma_j,\ j = 1, 2$, of $f(\mathbb{D})$, we must have that the image under the kernel Q of the preimage of one endpoint of Γ_j lies to the left of the vertical line l determined by $x = x_0$ and the image under the kernel Q of the preimage of other endpoint Γ_j lies to the right of l. See Figure 3.

We consider the vertex $z_0 \in \Gamma_1$ whose image under $Q \circ f^{-1}$ lies to the right of l. Apply the variation described in Lemma 2.3 at the vertex z_0 to add another side to $f(\mathbb{D})$, making sure the added side is short enough that its image under $Q \circ f^{-1}$ still lies completely to the right of l. At the same time, push the entire side Γ_2 out so that the mapping radius is preserved and $f_\epsilon \in H_\alpha^2$.

By the above variational argument, the newly varied function has a smaller value for L. But this means f cannot be extremal. Thus, the extremal function for L in H_α^n, $n \geq 3$, cannot have two proper sides. It follows therefore that the extremal function in H_α^n can have at most one proper side.

Since $H_\alpha^2 \subset H_\alpha^n$ for all $n \geq 3$, if f is extremal in H_α^n and is an element of H_α^2, it must be extremal in H_α^2 as well. Thus, the extremal element in H_α^2 has at most one proper side. As a result, the extremal value for L in each H_α^n is achieved by the region with at most one proper side and hence the extremal value for $H_\alpha = \overline{\cup_{n\in\mathbb{N}} H_\alpha^n}$ is achieved by a region with at most one proper side.

Finally, it is clear that the range of $k_\alpha(z)/z$ is symmetric about the real axis. It can be shown for fixed $r,\ 0 < r < 1$, that $\operatorname{Re}\{k_\alpha(re^{i\theta})/re^{i\theta}\}$ is a monotonically decreasing function of $\theta,\ 0 < \theta < \pi$. Hence, the minimum value of L over H_α is achieved at $-k_\alpha(-r)/r$, $r = |z|$. A similar argument shows that the maximum occurs at $k_\alpha(r)/r$.

3.2. Proof of Theorem 1.2. Next we fix $0 < \alpha \leq 1$ and employ a similar argument to show that the maximum value of

$$L(f) = \operatorname{Re}\ a_3$$

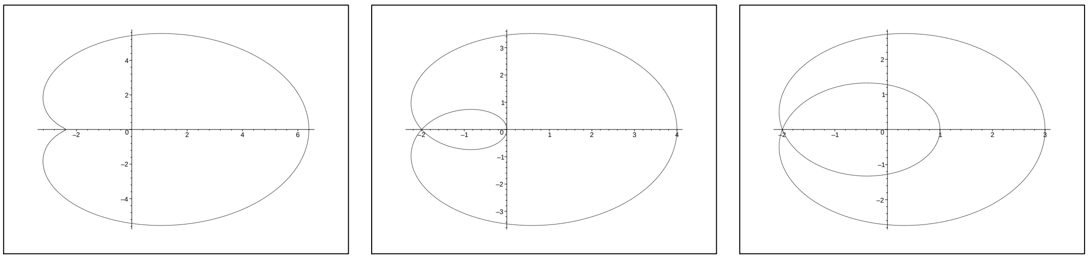

FIGURE 4. The cardioids resulting from $a_2 = 1.1$ (left), 0.5 (center), and 0.25 (right).

for a function $f(z) = \alpha z + a_2 z^2 + a_3 z^3 + \dots$ in H_α is obtained by a function mapping onto a domain with at most two sides. Suppose first that $f \in H_n^\alpha$ is extremal over H_n^α for some $n \geq 5$ and $f(\mathbb{D})$ has at least five proper sides Γ_j, $j = 1, \dots, 5$. Let γ_j be the preimage in $\partial\mathbb{D}$ of Γ_j, $j = 1, \dots, 5$.

If we apply the variation of Lemma 2.2 to each Γ_j with control $\epsilon_j = \epsilon\lambda_j$, then we produce a hyperbolically convex function defined by

$$f_\epsilon(z) = f(z) + \epsilon z f'(z) \sum_{j=1}^{5} \frac{\lambda_j}{2\pi} \int_{\gamma_j} \frac{1 + \xi z}{1 - \xi z}\, d\Psi + o(\epsilon).$$

Expanding $\frac{1+\xi z}{1-\xi z}$ as a series, we see

$$f_\epsilon(z) = f(z) + \epsilon z f'(z) \sum_{j=1}^{5} \frac{\lambda_j}{2\pi} \int_{\gamma_j} (1 + 2\xi z + 2\xi^2 z^2 + 2\xi^3 z^2 + \dots)\, d\Psi + o(\epsilon).$$

Now since $zf'(z) = \alpha z + 2a_2 z^2 + 3a_3 z^3 + \dots$, we have

$$f_\epsilon(z) = f(z) + \epsilon \sum_{j=1}^{5} \frac{\lambda_j}{2\pi} \int_{\gamma_j} (\alpha z + 2(a_2 + \alpha\xi) z^2 + (3a_3 + 4a_2\xi + 2\alpha\xi^2) z^3 + \dots)\, d\Psi + o(\epsilon).$$

Finally, gathering the powers of z, we arrive at

$$\begin{aligned} f_\epsilon(z) = {} & \alpha\left(1 + \epsilon \sum_{j=1}^{5} \frac{\lambda_j}{2\pi} \int_{\gamma_j} d\Psi\right) z + \left(a_2 + \epsilon \sum_{j=1}^{5} \frac{\lambda_j}{2\pi} \int_{\gamma_j} 2(a_2 + \alpha\xi)\, d\Psi\right) z^2 \\ & + \left(a_3 + \epsilon \sum_{j=1}^{5} \frac{\lambda_j}{2\pi} \int_{\gamma_j} (3a_3 + 4a_2\xi + 2\alpha\xi^2)\, d\Psi\right) z^3 + \dots + o(\epsilon). \end{aligned} \tag{10}$$

Consequently,

$$\frac{\partial}{\partial\epsilon} \operatorname{Re} L(f_\epsilon)\bigg|_{\epsilon=0} = \operatorname{Re} \sum_{j=1}^{5} \frac{\lambda_j}{2\pi} \int_{\gamma_j} (3a_3 + 4a_2\xi + 2\alpha\xi^2)\, d\Psi \tag{11}$$

and

$$\frac{\partial \Delta mr(f_\epsilon, f)}{\partial\epsilon}\bigg|_{\epsilon=0} = \sum_{j=1}^{5} \frac{\alpha\lambda_j}{2\pi} \int_{\gamma_j} d\Psi. \tag{12}$$

Now notice that the kernel $Q(\xi) = 3a_3 + 4a_2\xi + 2\alpha\xi^2$ of the integral in (11) maps the unit circle $\partial\mathbb{D}$ onto a cardioid. See Figure 4. The shape of the cardioid depends on a_2 and α, but a

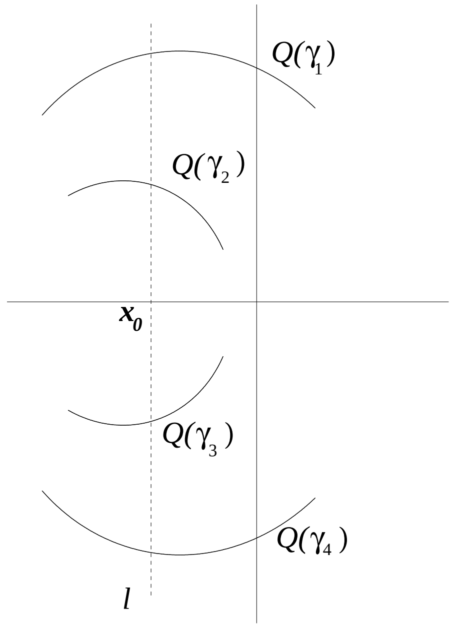

FIGURE 5. If the extremal function has four sides, then each curve $Q(\gamma_j)$ must cross the vertical line l given by $x = x_0 = \operatorname{Re} Q(\xi_j)$, $j = 1, \ldots, 4$.

simple argument shows that no vertical line intersects the cardioid more than four times. Thus if we apply the Mean Value Theorem to each of the integrals $\int_{\gamma_j} (3a_3 + 4a_2\xi + 2\alpha\xi^2)\, d\Psi$, we have

$$\operatorname{Re} \frac{\lambda_j}{2\pi} \int_{\gamma_j} (3a_3 + 4a_2\xi + 2\alpha\xi^2)\, d\Psi = \operatorname{Re} \frac{\lambda_j Q(\xi_j)}{2\pi} \int_{\gamma_j} d\Psi$$

for some $\xi_j \in \gamma_j$, $j = 1, \ldots, 5$.

Suppose the values of $\operatorname{Re} Q(\xi_j)$ are not the same for all j, say $\operatorname{Re} Q(\xi_1) < \operatorname{Re} Q(\xi_2)$. Then just as in the proof of Theorem 1.1, we can push Γ_1 in and Γ_2 out, preserving the mapping radius to produce a new map $f_\epsilon \in H_n^\alpha$ with

$$\frac{\partial}{\partial \epsilon} L(f_\epsilon)\bigg|_{\epsilon=0} > 0.$$

Consequently, since f is extremal, then $\operatorname{Re} Q(\xi_1) = \cdots = \operatorname{Re} Q(\xi_5)$. But that means the vertical line l determined by $x = x_0 = \operatorname{Re} Q(\xi_1) = \cdots = \operatorname{Re} Q(\xi_5)$ intersects the cardioid $Q(\partial\mathbb{D})$ in five distinct points. This contradiction implies f can have at most four sides.

Now notice that if f is an extremal function with exactly four sides, then as above, $\operatorname{Re} Q(\xi_j) = x_0$ for each $j = 1, \ldots, 4$, as otherwise we could vary two sides and increase the value of $L(f)$. Geometrically, this means that each curve $Q(\gamma_j)$ must cross the vertical line l. See Figure 5.

As a result, one endpoint of each curve $Q(\gamma_j)$ must lie to the left of l, and we can use the variation described in Lemma 2.3 to add a new side Γ_ϵ at the end of one of the current sides. If Γ_ϵ is sufficiently short and $\gamma_\epsilon = f^{-1}(\Gamma_\epsilon)$, then $Q(\gamma_\epsilon)$ will lie completely to the left of l. Thus there exists ξ_ϵ so that

$$\operatorname{Re} \int_{\gamma_\epsilon} Q(\xi)\, d\Psi = \operatorname{Re} Q(\xi_\epsilon) \int_{\gamma_\epsilon} d\Psi$$

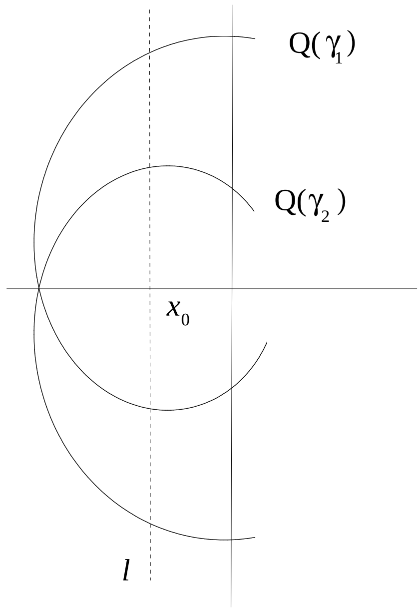

FIGURE 6. For a function with two sides, all four endpoints of $Q(\gamma_j)$ may lie on the same side of the line l determined by $x = x_0 = \operatorname{Re} Q(\gamma_1) = \operatorname{Re} Q(\gamma_2)$.

and

$$\operatorname{Re} Q(\xi_\epsilon) < x_0.$$

Then arguing as above, we produce a new function $f_\epsilon \in H_\alpha^5$ with five sides and $L(f_\epsilon) > L(f)$.

But the extremal function in H_α^5 has at most four sides. This contradiction means the extremal function in H_α^n, $n \geq 3$ can have at most three sides.

However, notice that if the extremal function has three sides, then we must still have one endpoint of $Q(\gamma_j)$ on the left of the line l for some $j = 1, 2, 3$. Thus the argument above can be repeated to show that f can have at most two sides.

On the other hand, if f has two sides, it is certainly possible for all four endpoints of $Q(\gamma_j)$, $j = 1, 2$ to lie to the right of l, precluding the reduction to only one side. See Figure 6. This is entirely to be expected, of course, as Ma and Minda have shown the extremal domain for $\alpha = 1/2$ has more than one side [7].

Since the extremal function in H_α^n for each $n \geq 2$ has at most two sides and $\cup_n H_\alpha^n$ is dense in H_α, the maximum value of $L(f)$ is obtained by function mapping onto a domain with at most two proper sides.

3.3. Proof of Theorem 1.3. In Theorem 1.3 we consider the same functional as in Theorem 1.2, but we maximize not only over all $f \in H_\alpha$, but also over all α. The arguments used in the proof above still apply, only we no longer need to ensure

$$\left.\frac{\partial \Delta mr(f_\epsilon, f)}{\partial \epsilon}\right|_{\epsilon=0} = \sum_{j=1}^{5} \frac{\lambda_j \alpha}{2\pi} \int_{\gamma_j} d\Psi = 0$$

when we perform a variation. In any case, we see there is an extremal function mapping onto a region with at most two sides.

References

[1] Roger W. Barnard and John L. Lewis. Subordination theorems for some classes of starlike functions. *Pacific J. Math.*, 56(2):333-366, 1975.

[2] Roger W. Barnard, Leah B. Cole, Kent Pearce, and G. Brock Williams. Sharp bounds for the Schwarzian derivative for hyperbolically convex functions. *Preprint*, 2003.

[3] Roger Barnard and John L. Lewis, *Subordination theorems for some classes of starlike functions*, Pacific J. Math. **56** (1975), no. 2, 333–366. MR52 #721

[4] Roger W. Barnard, G. Lenny Ornas, and Kent Pearce. A variational method for hyperbolically convex functions. *Preprint*, 2003.

[5] Roger W. Barnard and Glenn Schober, *Möbius transformations of convex mappings*, Complex Variables Theory Appl. **3** (1984), no. 1-3, 55–69. MR85j:30012

[6] A. Beardon, *Geometry of discrete groups*, Springer-Verlag, Berlin, 1983.

[7] William Ma and David Minda, Hyperbolically convex functions, *Ann. Polon. Math.* **60** (1994), no. 1, 81–100. MR**95k**:30037

[8] William Ma and David Minda, Hyperbolically convex functions. II, *Ann. Polon. Math.* **71** (1999), no. 3, 273–285. MR**2000j**:30020

[9] Diego Mejía and Christian Pommerenke, On hyperbolically convex functions, *J. Geom. Anal.* **10** (2000), no. 2, 365–378.

[10] Diego Mejía and Christian Pommerenke, On spherically convex univalent functions, *Michigan Math. J.* **47** (2000), no. 1, 163–172. MR**2001a**:30013

[11] Diego Mejía and Christian Pommerenke, Sobre la derivada Schawarziana de aplicaciones conformes hiperbólicamente, *Revista Colombiana de Matemáticas* **35** (2001), no. 2, 51–60.

[12] Diego Mejía and Christian Pommerenke, Hyperbolically convex functions, dimension and capacity, *Complex Var. Theory Appl.* **47** (2002), no. 9, 803–814. MR**2003f**:30017

[13] Diego Mejía and Christian Pommerenke, On the derivative of hyperbolically convex functions, *Ann. Acad. Sci. Fenn. Math.* **27** (2002), no. 1, 47–56.

[14] Diego Mejía, Christian Pommerenke, and Alexander Vasil′ev, Distortion theorems for hyperbolically convex functions, *Complex Variables Theory Appl.* **44** (2001), no. 2, 117–130. MR**2003e**:30025

[15] Michalska, Malgorzata, Dmitri V. Prokhorov, and Jan Szynal, The compositions of hyperbolic triangle mappings, *Complex Variables Theory Appl.* **43** (2000), no. 2, 179-186. MR **2001m**:30008

[16] Christian Pommerenke, private communication.

[17] A. Yu. Solynin, Some extremal problems on the hyperbolic polygons, *Complex Variables Theory Appl.* **36** (1998), no. 3, 207–231. MR**99j**:30030

[18] A. Yu. Solynin, Moduli and extremal metric problems, *Algebra i Analiz* **11** (1999), no. 1, 3–86, translation in *St. Petersburg Math. J.* **11** (2000), no. 1, 1–65. MR**2001b**:30058

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