An isoperimetric inequality for logarithmic capacity

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Abstract

We prove a sharp lower bound of the form $\operatorname{cap} E \geq (1/2)\operatorname{diam} E \cdot \Psi(\operatorname{area} E/((\pi/4)\operatorname{diam}^2 E))$ for the logarithmic capacity of a compact connected planar set E in terms of its area and diameter. Our lower bound includes as special cases G. Faber's inequality $\operatorname{cap} E \geq \operatorname{diam} E/4$ and G. Pólya's inequality $\operatorname{cap} E \geq (\operatorname{area} E/\pi)^{1/2}$. We give explicit formulations, functions of $(1/2)\operatorname{diam} E$, for the extremal domains which we identify. [1] [2]

1 Introduction

The logarithmic capacity, $\operatorname{cap} E$, of a continuum (= compact connected set) E in the complex plane $\mathbb{C}$ is defined by

$$-\log \operatorname{cap} E = \lim_{z\to\infty} (g(z) - \log|z|), \tag{1.1}$$

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where $g(z)$ denotes Green's function of the unbounded component $\Omega(E)$ of $\overline{\mathbb{C}} \setminus E$ having singularity at $z = \infty$.

The measure of a set described by the logarithmic capacity is very important in potential theory, analysis, and PDE's. It combines several characteristics of a compact set, among which are the geometric concept of transfinite diameter due to M. Fekete, the concept of Chebyshev's constant from polynomial approximation, and the concept of the outer radius from conformal mapping, see [D, Du, G, H].

In general, computation of $\operatorname{cap} E$ is a difficult problem but there are several estimates of $\operatorname{cap} E$ in terms of geometric characteristics of E that are very useful in applications, see [PSz]. For instance,

$$(1/4)\operatorname{diam} E \leq \operatorname{cap} E \leq (1/2)\operatorname{diam} E, \tag{1.2}$$

$$((1/\pi)\operatorname{area} E)^{1/2} \leq \operatorname{cap} E < \infty. \tag{1.3}$$

The first inequality in (1.2) was found by G. Faber [F] in a different form. The inequalities in (1.3) and the second inequality in (1.2), which is valid for any (not necessary connected) compact set, were proved by G. Pólya [Po]. Equality occurs only for rectilinear segments in Faber's inequality and for disks in Pólya's inequalities. The case of equality in the right inequality in (1.2) was studied by J. Jenkins [J] and A. Pfluger [Pf].

We will employ the following notation throughout this paper: let $\mathbb{U} = \{z : |z| < 1\}$ and $\mathbb{U}_r(c) = \{z : |z - c| < r\}$, so that $\mathbb{U} = \mathbb{U}_1(0)$. Finally, let $\mathbb{U}^* = \overline{\mathbb{C}} \setminus \overline{\mathbb{U}}$.

In this paper, we prove the following theorem that contains the left inequalities in (1.2) and (1.3) as special cases.

Theorem 1 *Let E be a continuum in $\mathbb{C}$. Then*

$$\operatorname{cap} E \geq (1/2)\operatorname{diam} E \cdot \Psi(\operatorname{area} E/((\pi/4)\operatorname{diam}^2 E)), \tag{1.4}$$

where $1/\Psi(s)$ is a decreasing function from $[0,1]$ onto $[1,2]$ that is the inverse function to $s = p^{-2}[\beta^2(p) - 2p(\beta(p) - 1)]$, with $1 \leq \beta(p) \leq 2$ defined by equation (1.7), with d replaced by p.

Equality occurs in (1.4) if and only if $E = a(\mathbb{C} \setminus f_d(\mathbb{U}^*)) + b$ for some $a, b \in \mathbb{C}$, $a \neq 0$ and $1 \leq d \leq 2$, where the function $f_d(z)$ is defined for $|z| > 1$ by (1.8).

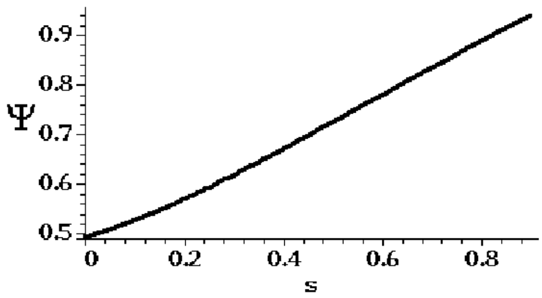

Figure 1: Graph of $\Psi(s)$

The graph of Ψ is plotted in Figure 1. Figure 2 displays the shape of the extremal continua for $a = 1$, $b = 0$ and some typical values of d.

Let $s = \operatorname{area} E/((\pi/4)\operatorname{diam}^2 E)$. In the case $s = 0$, (1.4) gives Faber's inequality and in the case $s = 1$, it gives Pólya's inequality. Combined with the right-hand side inequality in (1.2) and the classical area-diameter inequality $0 \leq \operatorname{area} E/((\pi/4)\operatorname{diam}^2 E) \leq 1$, (1.4) describes the range of one of the quantities $\operatorname{cap} E$, $\operatorname{diam} E$, or $\operatorname{area} E$ if the other two are fixed. Several similar sharp inequalities linking three characteristics of a set are known in geometry. But to prove such a sharp inequality is a difficult task even for purely geometric quantities, see [JBo]. From this perspective, (1.4) might be the first known sharp inequality for these three quantities that includes a functional characteristic.

Since $\operatorname{cap} E$, $\operatorname{diam} E$, and $(\operatorname{area} E)^{1/2}$ all change linearly with respect to scaling we can fix one of them, say $\operatorname{cap} E$, and then study the region of variability of the other two. In this way, we can reformulate the problem as finding the maximal omitted area for the class Σ of univalent functions

$$f(z) = z + a_0(f) + a_1(f)z^{-1} + \dots \tag{1.5}$$

which are analytic in $\mathbb{U}^*$, except for a simple pole at $z = \infty$. For $f \in \Sigma$, let $E_f = \mathbb{C} \setminus f(\mathbb{U}^*)$. It is well known, as a consequence of the normalization in (1.5), that for $f \in \Sigma$ that $1 \leq (1/2)\operatorname{diam} E_f \leq 2$. Therefore, for $1 \leq d \leq 2$, we will consider $\Sigma_d = \{f \in \Sigma : \operatorname{diam} E_f = 2d\}$. For $f \in \Sigma_d$, define $A_f = \operatorname{area} E_f$

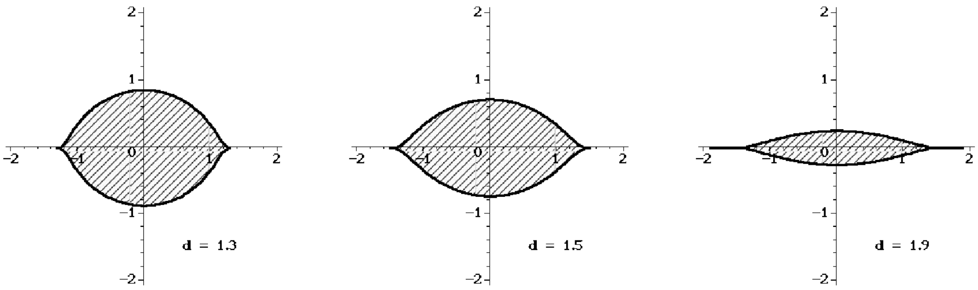

Figure 2: Extremal continua

and $A(d) = \sup_{f\in\Sigma_d} A_f$. It is well known that

$$A_f = \pi(1 - \sum_{n=1}^{\infty} n|a_n(f)|^2)$$

— this relation will be often used in what follows. Theorem 1 is equivalent to

Theorem 2 *Let $f \in \Sigma_d$, $1 \le d \le 2$. Then,*

$$A(f) \le \pi[\beta^2 - 2d(\beta - 1)], \tag{1.6}$$

where $1 \le \beta \le 2$ is the unique solution to the equation

$$d = \beta - (\beta - 1)\log(\beta - 1). \tag{1.7}$$

Equality occurs in (1.6) if and only if $f(z) = e^{i\tau} f_d(e^{-i\tau} z) + b$ with $\tau \in \mathbb{R}$, $b \in \mathbb{C}$, and

$$f_d(z) = d + \int_1^z z^{-1}\varphi(z; d)\, dz, \tag{1.8}$$

where the function $\varphi(z; d) = z f_d'(z)$ is defined by the equation

$$\varphi(z; d) = A\frac{z^2 - 1}{z}\,\frac{\sqrt{1 + Bz^2 + \sqrt{r(z)}}\sqrt{B + z^2 + \sqrt{r(z)}}}{c(1 + z^2) + \sqrt{r(z)}} \tag{1.9}$$

with principal branches of the radicals and

$$c = \beta - 1 \ , \qquad A \ = \ \frac{1+c}{2c} \ , \qquad B = 2c^2 - 1 \ ,$$
$$r(z) \ = \ 1 + 2Bz^2 + z^4 \ .$$

The function φ maps $\mathbb{U}^*$ conformally onto the complement of the "double anchor"

$$F(\beta, \psi) = [-i\beta, i\beta] \cup \{\beta e^{it} : \frac{\pi}{2} - \psi \le t \le \frac{\pi}{2} + \psi\}$$

$$\cup \{\beta e^{it} : \frac{3\pi}{2} - \psi \le t \le \frac{3\pi}{2} + \psi\},$$

where β is defined by (1.7) and $\psi = (1/2)\cos^{-1}(8\beta^{-1} - 8\beta^{-2} - 1)$.

The graph of the maximal omitted area $A(d) = \pi[\beta^2(d) - 2d(\beta(d) - 1)]$ is shown in Figure 3.

To prove Theorem 2, we apply techniques developed in [BS], which were based on symmetrization transformations and some elementary local variations. Section 2 contains preliminary results and necessary definitions. In Section 3, we identify the extremal function by solving a specific boundary value problem for analytic functions. Section 4 completes the proofs of Theorems 1 and 2.

Note that some similar sharp estimates for the area of $f(\mathbb{U})$ for problems with analytic side conditions instead of geometric constraints, as imposed in the present paper, were found in [ASS1, ASS2] using a different method.

2 Preliminaries

First we show the existence of an extremal function and describe some simple properties of the maximal omitted area.

Lemma 1 *For every $1 \le d \le 2$ there is a function $f \in \Sigma_d$ such that $A_f = A(d)$.*

The maximal area $A(d)$ is continuous and strictly decreases from π to 0 as d increases from 1 to 2.

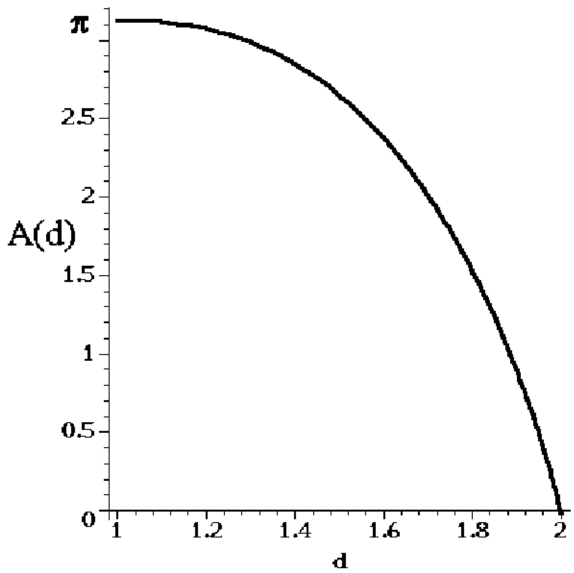

Figure 3: Graph of $A(d)$

Proof. For a fixed d, Σ_d is compact in the topology of uniform convergence on compact subsets of $\mathbb{U}^*$. Since A_f is upper semi-continuous, the existence of an extremal function follows.

Let $1 < d_1 < d_2 \leq 2$ and let $f \in \Sigma_d$ be extremal for $A(d_2)$. Note that f has at least one non-zero coefficient $a_k(f)$ for some $k \geq 1$. The function

$$f_t(z) = t^{-1} f(tz) = z + a_0(t)t^{-1} + a_1(t)t^{-2}z^{-1} + \dots, \tag{2.1}$$

as well as the area A_{f_t} and diameter $d(f_t) = \operatorname{diam} E_{f_t}$ depend continuously on t, $1 \leq t < \infty$. Since $f_t(z) \to z$ as $t \to \infty$, one can easily show that $d(f_t) \to 2$ as $t \to \infty$. Hence, there is $t_1 > 1$ such that $d(f_{t_1}) = 2d_1$. Therefore

$$A(d_1) \geq A_{f_{t_1}} = \pi(1 - \sum_{n=1}^{\infty} n t_1^{-2(n+1)} |a_n(f)|^2) > \pi(1 - \sum_{n=1}^{\infty} n |a_n(f)|^2) = A(d_2), \tag{2.2}$$

with strict inequality since $a_k(f) \neq 0$. Equation (2.2) proves the strict monotonicity of $A(d)$.

Finally, the compactness of Σ_d and continuity of A_{f_t} imply the continuity of $A(d)$. □

Since the class Σ_d is invariant under the rigid motions of $\mathbb{C}$, i.e. $e^{-i\theta} f(e^{i\theta} z) + b \in \Sigma_d$ if $f \in \Sigma_d$ and $\theta \in \mathbb{R}$, $b \in \mathbb{C}$, we may restrict ourselves to functions $f \in \Sigma_d$ such that the points $w_1 = d$, $w_2 = -d$ belong to E_f. Thus, the condition $\operatorname{diam} E_f = |w_1 - w_2| = 2d$ will be assumed if a different condition is not imposed explicitly.

To prove symmetry properties of an extremal continuum E_f we shall apply Steiner symmetrization defined as follows:

The Steiner symmetrization of a compact set E w.r.t. the real axis $\mathbb{R}$ is a compact set E^* such that for every $u \in \mathbb{R}$, $E^* \cap l(u) = \emptyset$ if $E \cap l(u) = \emptyset$ and $E^* \cap l(u) = \{w = u + it : -m \le t \le m\}$ if $E \cap l(u) \neq \emptyset$. Here $l(u) = \{w = u + it : -\infty < t < \infty\}$ and $m = \operatorname{meas}(E \cap l(u))$ denotes the linear Lebesgue measure. Steiner symmetrization w.r.t. the imaginary or other axis is defined similarly.

It is well known that Steiner symmetrization preserves area and diminishes diameter and logarithmic capacity [H, D].

Lemma 2 *For $1 < d < 2$, let $f \in \Sigma_d$ be an extremal function normalized as above. Then, E_f possesses Steiner symmetry w.r.t. the real and imaginary axes. Moreover, the boundary of E_f, ∂E_f, consists of a Jordan rectifiable curve L_f plus, possibly, some added segments $I_+ = [d_0, d]$, $I_- = [-d, -d_0]$, $0 < d_0 = d_0(d) \le d$, of the real axis.*

Proof. Suppose that E_f does not possess Steiner symmetry w.r.t. $\mathbb{R}$. Let E^* be the Steiner symmetrization of E_f w.r.t. $\mathbb{R}$. Note that

$$2d = \operatorname{diam} E_f = \operatorname{diam} E^* \quad \text{and} \quad \operatorname{cap} E_f > \operatorname{cap} E^* \tag{2.3}$$

since the points $\pm d \in E_f$ and since E^* is not a rigid motion of E_f (see [D]). Let

$$F(z) = \alpha z + \alpha_0 + \alpha_1 z^{-1} + \ldots, \quad \alpha > 0, \tag{2.4}$$

map $\mathbb{U}^*$ conformally onto $\Omega(E^*)$. The inequality in (2.3) shows that $\alpha < 1$. Let $F_\alpha = \alpha^{-1} F$. Then, $F_\alpha \in \Sigma_{d/\alpha}$. Therefore, we have

$$A(d/\alpha) \ge A_{F_\alpha} = \pi\alpha^{-2}(1 - \sum_{n=1}^{\infty} n\alpha^{-2}|\alpha_n|^2) \ge \pi(1 - \sum_{n=1}^{\infty} n|a_n(f)|^2) = A(d).$$

Since $d/\alpha > d$, the latter contradicts the strict monotonicity of $A(d)$ in Lemma 1.

The same arguments show that E_f possesses Steiner symmetry w.r.t. the imaginary axis.

Let $L_f^+ = \{w \in \partial E_f : \Im w > 0\}$ and $L_f^- = \{w \in \partial E_f : \Im w < 0\}$. The Steiner symmetries of E_f w.r.t. the real and imaginary axes can be used to show that L_f^+ and L_f^- are rectifiable Jordan arcs; a similar argument was used in [ASS2, Lemma 4]. Indeed, let $L^+ = \{w \in L_f^+ : \Re w > 0\}$ and let $d_0 = \sup\{\Re w : w \in L^+\}$, $m_0 = \sup\{\Im w : w \in L^+\}$. The function

$$\tau(w) = u + m_0 - v, \quad \text{where} \quad w = u + iv$$

is continuous on L^+ and maps the closure $\bar{L}^+$ one-to-one onto the segment $\{\tau : 0 \leq \tau \leq d_0 + m_0\}$. Therefore, $\bar{L}^+$ is Jordan. Since $\Re w$ and $\Im w$ are both monotonic on L^+, it follows that $\bar{L}^+$ is rectifiable. This implies that ∂E_f consists of a rectifiable Jordan curve L_f plus, possibly, some added horizontal segments $[-d, -d_0]$, $[d_0, d]$ and vertical segments $[-im, -im_0]$, $[im_0, im]$ with $0 \leq m_0 \leq m < \infty$.

The presence of vertical segments, i.e., the segments $[-im, -im_0]$, $[im_0, im]$ with $m > m_0$ easily leads to a contradiction: shortening the vertical slits and expanding the horizontal ones we can find a continuum $\tilde{E}$ such that $\operatorname{area} \tilde{E} = \operatorname{area} E_f$, $\operatorname{cap} \tilde{E} = \operatorname{cap} E_f = 1$, and $\operatorname{diam} \tilde{E} > \operatorname{diam} E_f$ that contradicts the strict monotonicity property of $A(d)$ in Lemma 1. $\Box$

Let $f \in \Sigma_d$ be an extremal function for $A(d)$. By Lemma 2, $\partial f(\mathbb{U}^*) = L_f^+ \cup L_f^- \cup [-d, -d_0] \cup [d_0, d]$ with $0 < d_0 \leq d$. If $d_0 < d$ then there is $0 < \theta_0 < \pi/2$ such that $L_f^+ = \{f(z) : z \in l_f^+\}$, $[d_0, d] = \{f(z) : z \in l_n^{++}\}$, $[-d, -d_0] = \{f(z) : z \in l_n^{+-}\}$, where $l_f^+ = \{e^{i\theta} : \theta_0 < \theta < \pi - \theta_0\}$, $l_n^{++} = \{e^{i\theta} : 0 \leq \theta \leq \theta_0\}$, $l_n^{+-} = \{e^{i\theta} : \pi - \theta_0 \leq \theta \leq \pi\}$. The corresponding arcs in the lower half-plane will be denoted by l_f^-, l_n^{-+}, and l_n^{--}. The image curves L_f^+ and L_f^- are called the *free boundary*, the preimages l_f^+ and l_f^- are called the *free arcs*. Respectively, $[d_0, d]$, $[-d, -d_0]$ and l_n^{++}, l_n^{+-}, l_n^{-+}, l_n^{--} are called the *non-free boundary* and the *non-free arcs*.

To study the behavior of f' on the non-free arcs we shall use two lemmas from [BS], which are limiting cases of Theorem 1 in [S2].

Let H_τ^+ and H_τ^- be the left and right half-planes w.r.t. the vertical line $l(\tau) = \{w = u + iv : u = \tau\}$. For $D \subset \overline{\mathbb{C}}$, let $D_\tau^+ = D \cap H_\tau^+$, $D_\tau^- = D \cap H_\tau^-$ and let D_τ^* denote the set symmetric to D w.r.t. $l(\tau)$, i.e. $D_\tau^* = \{w = u + iv : 2\tau - u + iv \in D\}$.

We say that D possesses the *polarization property* in the interval $\tau_1 < \tau < \tau_2$ if $(D_\tau^-)^*_\tau \subset D_\tau^+$ for all $\tau_1 < \tau < \tau_2$. Extremal configurations shown in Figure 2 give an example of domains possessing the polarization property in the corresponding intervals $0 < \tau < d$.

Lemma 3 *[BS, Lemma 4] Let $f \in \Sigma$, $D = f(\mathbb{U}^*)$, and let f map a boundary arc $\{e^{i\theta} : \theta_1 < \theta < \theta_2\}$ onto a horizontal interval $\{w : \Im w = v_0, \tau_1 < \Re w < \tau_2\}$. Let D possess the polarization property in $\tau_1 < \tau < \tau_2$. Then, $|f'(e^{i\theta})|$ strictly increases in $\theta_1 < \theta < \theta_2$ if $f(e^{i\theta_1}) = \tau_2 + iv_0$ and strictly decreases if $f(e^{i\theta_1}) = \tau_1 + iv_0$.*

If f is extremal for $A(d)$, the joint symmetry of E_f assures that the polarization property of $D = f(\mathbb{U}^*)$ holds in $d_0 < \tau < d$. Thus, we obtain from Lemma 3,

Corollary 1 *Let $f \in \Sigma_d$ be extremal for $A(d)$, $1 < d < 2$ and suppose that the non-free arc l_n^{++} is not degenerate, i.e., $0 < \theta_0 < \pi/2$. Then, $|f'(e^{i\theta})|$ strictly increases in $0 < \theta < \theta_0$ and $\pi < \theta < \pi + \theta_0$ and strictly decreases in $\pi - \theta_0 < \theta < \pi$ and $2\pi - \theta_0 < \theta < 2\pi$.*

We need a similar result concerning angular polarization. Let $\gamma_\varphi = \{w = te^{i\varphi}, t \geq 0\}$ and let H_φ^+ denote the right half-plane w.r.t. the line determined by γ_φ and H_φ^- denote the left half-plane w.r.t. the line determined by γ_φ. For $D \subset \overline{\mathbb{C}}$, let $D_\varphi^+ = D \cap H_\varphi^+$, $D_\varphi^- = D \cap H_\varphi^-$ and let D_φ^* denote the set symmetric to D w.r.t. the line determined by γ_φ .

We say that a domain D possesses the *angular polarization property* in $\varphi_1 < \varphi < \varphi_2$ if $(D_\varphi^-)^*_\varphi \subset D_\varphi^+$ for all $\varphi_1 < \varphi < \varphi_2$. For example, domain G depicted in Figure 4 possesses the angular polarization property in $0 < \varphi < \pi/2$.

Lemma 4 *[BS, Lemma 5] Let g map $\mathbb{U}^*$ conformally onto D and map a boundary arc $\{e^{i\theta} : \theta_1 < \theta < \theta_2\}$ onto a circular arc $L = \{w = \rho e^{i\varphi} : \varphi_1 < \varphi < \varphi_2\}$. Let $g(\infty) \in H_{\varphi_1}^+ \cap H_{\varphi_2}^+$ and let D possesses the angular polarization property in $\varphi_1 < \varphi < \varphi_2$. Then, $|g'(e^{i\theta})|$ strictly increases in $\theta_1 < \theta < \theta_2$ if $g(e^{i\theta_1}) = \rho e^{i\varphi_1}$ and strictly decreases in $\theta_1 < \theta < \theta_2$ if $g(e^{i\theta_1}) = \rho e^{i\varphi_2}$.*

Remark. The domains D in Lemmas 3 and 4 possess the polarization property for a horizontal interval and the angular polarization property w.r.t.

rays issuing from the origin, resp. One can easily reformulate these lemmas for arbitrary intervals and for rays issuing from arbitrary centers. For instance, Lemma 4 in [BS] is formulated for vertical intervals.

The term "polarization property" comes from the proofs of Theorem 1 in [S2] and Lemmas 4 and 5 in [BS] that use the polarization transformation.

To find a boundary condition for an extremal function $f \in \Sigma_d$ on the free arcs, we apply, in a suitable form, the local variation used in [BS]. First, we recall the relations linking the logarithmic capacity of a continuum E with the outer radius and reduced module of $\Omega(E)$. Let

$$g(w) = w + b_0 + b_1 w^{-1} + \dots$$

map $\Omega(E)$ conformally onto $\mathbb{U}_R^* = \overline{\mathbb{C}} \setminus \overline{\mathbb{U}}_R$, where $\mathbb{U}_R = \{\zeta : |\zeta| < R\}$. The radius $R = R(E)$ of the omitted disk is uniquely determined and is called the outer radius of $\Omega(E)$; it is well known that $\operatorname{cap} E = R(E)$, see [Du, § 10.2], [G, Ch. 7]. The quantity

$$m(\Omega(E), \infty) = -\frac{1}{2\pi} \log R(E) = -\frac{1}{2\pi} \log \operatorname{cap} E \tag{2.5}$$

is called the reduced module of $\Omega(E)$ at $w = \infty$.

The variation used in [BS] and in the present paper is rather complicated. Let $\Omega \subset \overline{\mathbb{C}}$ be a simply connected domain which contains ∞ such that its boundary arcs lying in the vicinities of two of its boundary points, w_1 and w_2, are Jordan and rectifiable. Let $\partial\Omega$ have a tangent l at w_1 and let n_1 be a unit inward normal at w_1. For $\varepsilon_1 > 0$ small enough, let $c^0_{\varepsilon_1}$ and c_{ε_1} be open and closed crosscuts of D at the boundary point w_1, i.e. $c^0_{\varepsilon_1}$ and c_{ε_1} are respectively the biggest open and closed arcs of $C_{\varepsilon_1}(w_1)$, where $C_r(w_0) = \{w : |w - w_0| = r\}$, such that $w_1 + \varepsilon_1 n_1 \in c^0_{\varepsilon_1} \subset \Omega$ and $w_1 + \varepsilon_1 n_1 \in c_{\varepsilon_1} \subset \bar{\Omega}$, respectively. Let $\mathbb{U}^+_{\varepsilon_1}(w_1)$ denote the connected component (half-disk) of $\mathbb{U}_{\varepsilon_1}(w_1) \setminus l$ that contains the point $w_1 + \varepsilon_1 n_1$ on its boundary. Let c'_{ε_1} denote the maximal open circular arc contained in the intersection $c^0_{\varepsilon_1} \cap \partial\mathbb{U}^+_{\varepsilon_1}(w_1)$. Let $\hat{\Omega}_{\varepsilon_1}$ be a connected component of $\Omega \setminus \overline{\mathbb{U}}_{\varepsilon_1}(w_1)$ containing ∞ and let

$$\Omega_{\varepsilon_1} = \hat{\Omega}_{\varepsilon_1} \cup \mathbb{U}^+_{\varepsilon_1}(w_1) \cup c'_{\varepsilon_1}. \tag{2.6}$$

Let $I(\varepsilon_1) = \{w = w_1 - itn_1 : -\varepsilon_1 < t < \varepsilon_1\}$. For $0 < \varphi_1 \le 1/2$, let $M(\varepsilon_1, \varphi_1)$ denote the open lune in $\mathbb{U}_{\varepsilon_1}(w_1) \setminus \mathbb{U}^+_{\varepsilon_1}(w_1)$ bounded by $I(\varepsilon_1)$ and a circular arc $\gamma(\varepsilon_1, \varphi_1)$ that forms angles of opening $\pi\varphi_1$ with the interval $I(\varepsilon_1)$ at its end points. Let

$$\Omega(\varepsilon_1, \varphi_1) = \Omega_{\varepsilon_1} \cup M(\varepsilon_1, \varphi_1) \cup I(\varepsilon_1). \tag{2.7}$$

Let $g(w) = g(w; \varepsilon_1, \varphi_1)$ map $\Omega(\varepsilon_1, \varphi_1)$ conformally onto $\mathbb{U}^*$ such that $g(\infty) = \infty$, $g(w_2) = 1$. Let $0 < \varphi_2 \le 1/2$ and $\varepsilon_2 > 0$ be small enough. Let $U^{\varepsilon_2, \varphi_2} \subset \mathbb{U}^*$ be the simply connected domain which contains ∞ and which is bounded by the arc $L(\varepsilon_2) = \{e^{i\theta} : \varepsilon_2 \le |\theta| \le \pi\}$ and by the circular arc $L(\varepsilon_2, \varphi_2)$ with ends at the points $e^{i\varepsilon_2}$ and $e^{-i\varepsilon_2}$ that forms an angle of opening $\pi(1 - \varphi_2)$ with the arc $L(\varepsilon_2)$ at the points $e^{i\varepsilon_2}$ and $e^{-i\varepsilon_2}$.

The domain

$$\tilde{\Omega} = g^{-1}(U^{\varepsilon_2, \varphi_2}, \varepsilon_1, \varphi_1) \tag{2.8}$$

will be called *the two point variation of* Ω *centered at* w_1 *and* w_2 *with radii* ε_1 *and* ε_2 *and inclinations* φ_1 *and* φ_2.

The following lemma is a reformulation of Lemma 10 in [BS] for conformal mappings f of $\mathbb{U}^*$ normalized by condition $f(\infty) = \infty$; in [BS] this result is formulated for conformal mappings f of the unit disk $\mathbb{U}$ with normalization $f(0) = 0$.

Lemma 5 *[BS, Lemma 10] Let* $w = f(z)$ *map* $\mathbb{U}^*$ *conformally onto* Ω *defined above such that* $f(\infty) = \infty$, $f(e^{i\theta_1}) = w_1$, $f(e^{i\theta_2}) = w_2$ *and let there exist the limits*

$$f'(e^{i\theta_k}) = \lim_{z \to e^{i\theta_k}, z \in \overline{\mathbb{U}}^*} \frac{f(z) - w_k}{z - e^{i\theta_k}} \ne 0, \infty \quad \textit{for} \quad k = 1, 2. \tag{2.9}$$

Let $|f'(e^{i\theta_k})| = \alpha_k$, $k = 1, 2$. *Let* $\tilde{\Omega}(\varepsilon_1, \varepsilon_2, \varphi_1, \varphi_2)$ *be the two point variation of* Ω *defined by (2.8) with* ε_2 *replaced by* ε_2/α_2. *Then, for fixed* $0 < \varphi_1 \le 1/2$ *and* $0 < \varphi_2 \le 1/2$,

$$m(\tilde{\Omega}(\varepsilon_1, \varepsilon_2, \varphi_1, \varphi_2), \infty) - m(\Omega, \infty) =$$

$$\frac{\varphi_1(2 + \varphi_1)}{12\pi\alpha_1^2(1 + \varphi_1)^2}\varepsilon_1^2 - \frac{\varphi_2(2 - \varphi_2)}{12\pi\alpha_2^2(1 - \varphi_2)^2}\varepsilon_2^2 + o(\varepsilon_1^2) + o(\varepsilon_2^2) \tag{2.10}$$

and

$$\mathrm{area}\,(\mathbb{C} \setminus \tilde{\Omega}(\varepsilon_1, \varepsilon_2, \varphi_1, \varphi_2)) - \mathrm{area}\,(\mathbb{C} \setminus \Omega) =$$

$$-\frac{2\pi\varphi_1 - \sin 2\pi\varphi_1}{2\sin^2 \pi\varphi_1}\varepsilon_1^2 + \frac{2\pi\varphi_2 - \sin 2\pi\varphi_2}{2\sin^2 \pi\varphi_2}\varepsilon_2^2 + o(\varepsilon_1^2) + o(\varepsilon_2^2) \tag{2.11}$$

as $\varepsilon_1 \to 0$ *and* $\varepsilon_2 \to 0$.

To prove uniqueness of the extremal function in Σ_d, we need the following modification of Lemma 1 in [S1] where a similar result is proved for domains with one axis of symmetry.

Lemma 6 *(cf. [S1, Lemma 1]). For $k = 1, 2$, let $\Omega_k \subset \overline{\mathbb{C}}$ be simply connected domains which contain ∞ and which have double symmetry w.r.t. the coordinate axes. Let there be a point $\zeta \in \partial\Omega_1$, $\Re\zeta \geq 0$, $\Im\zeta > 0$ such that the points ζ, $\bar{\zeta}$, $-\zeta$, and $-\bar{\zeta}$ split $\partial\Omega_1$ into four boundary arcs l_r^+, l_r^-, l_i^+, and l_i^-, where l_r^+ lies in the closed right half-plane and connects ζ and $\bar{\zeta}$, l_i^+ lies in the closed upper half-plane and connects ζ and $-\bar{\zeta}$. Let*

$$g_k(z) = z + a_1(g_k)z^{-1} + \ldots$$

map $\mathbb{U}^*$ conformally onto Ω_k.

If $l_r^+ \subset \overline{\Omega}_2$, $l_i^+ \subset \mathbb{C} \setminus \Omega_2$, then

$$g_1(r) \leq g_2(r), \quad |g_2(ir)| \leq |g_1(ir)| \tag{2.12}$$

for all $r > 1$. Equality can occur in (2.12) if and only if $\Omega_1 = \Omega_2$.

3 Boundary value problem for extremal functions

In this section f will denote the extremal function in Σ_d with $1 < d < 2$ and $D = f(\mathbb{U}^*)$.

Lemma 7 *There is $\beta > 0$ such that*

$$|f'(e^{i\theta})| = \beta \quad \text{for a.e.} \quad e^{i\theta} \in l_f := l_f^+ \cup l_f^- \tag{3.1}$$

and

$$|f'(e^{i\theta})| < \beta \quad \text{for all} \quad e^{i\theta} \in l_n := \mathbb{T} \setminus \bar{l}_f. \tag{3.2}$$

Proof. Since $\partial D = L_f \cup I_+ \cup I_-$ by Lemma 2, where L_f is Jordan and rectifiable, the non-zero finite limit

$$f'(\zeta) = \lim_{z \to \zeta, z \in \overline{\mathbb{U}}^*} \frac{f(z) - f(\zeta)}{z - \zeta} \neq 0, \infty \tag{3.3}$$

exists for almost all $\zeta \in \mathbb{T}$. This easily follows from Theorem 6.8 in [P] applied to the univalent function $1/f(1/z)$. Assume that

$$0 < \beta_1 = |f'(e^{i\theta_1})| < |f'(e^{i\theta_2})| = \beta_2 < \infty \tag{3.4}$$

for some $e^{i\theta_1}, e^{i\theta_2} \in l_f$. Note that (3.3) and (3.4) allow us to apply the two point variation of Lemma 5.

For positive k_1, k_2 such that

$$0 < k_1 < 1 < k_2 \quad \text{and} \quad k_1\beta_1^{-1} > k_2\beta_2^{-1} \tag{3.5}$$

and for fixed $\varphi > 0$ small enough consider the two point variation $\tilde{D}$ of D centered at $w_1 = f(e^{i\theta_1})$ and $w_2 = f(e^{i\theta_2})$ with inclinations φ and radii $\varepsilon_1 = k_1\varepsilon$, $\varepsilon_2 = k_2\varepsilon$, respectively. Computing the change in the omitted area by formula (2.11), we find

$$\operatorname{area}(\mathbb{C}\setminus\tilde{D}) - \operatorname{area}(\mathbb{C}\setminus D) = \frac{2\pi\varphi - \sin 2\pi\varphi}{2\sin^2\pi\varphi}\varepsilon^2(k_2^2 - k_1^2) + o(\varepsilon^2).$$

Therefore,

$$\operatorname{area}(\mathbb{C}\setminus\tilde{D}) > \operatorname{area}(\mathbb{C}\setminus D) \tag{3.6}$$

for all $\varepsilon > 0$ small enough. Applying the variation (2.10) of Lemma 6, we get

$$\begin{aligned} m(\tilde{D},\infty) - m(D,\infty) &= \frac{1}{12\pi}\left[\frac{\varphi(2+\varphi)}{(1+\varphi)^2}\frac{k_1^2}{\beta_1^2} - \frac{\varphi(2-\varphi)}{(1-\varphi)^2}\frac{k_2^2}{\beta_2^2}\right]\varepsilon^2 + o(\varepsilon^2) \\ &= \left[\frac{\varphi}{6\pi}\left(\frac{k_1^2}{\beta_1^2} - \frac{k_2^2}{\beta_2^2}\right) + o(\varphi)\right]\varepsilon^2 + o(\varepsilon^2), \end{aligned} \tag{3.7}$$

which together with (3.5) implies that

$$m(\tilde{D},\infty) > m(D,\infty) \tag{3.8}$$

for all $\varepsilon > 0$ small enough if φ is chosen such that the expression in the brackets in (3.7) is positive. Let $E = \mathbb{C}\setminus D$, $\tilde{E} = \mathbb{C}\setminus\tilde{D}$. Equations (3.8) and (2.5) show that

$$\operatorname{cap}\tilde{E} < \operatorname{cap} E. \tag{3.9}$$

Since $\operatorname{area}\tilde{E} > \operatorname{area} E$ and $\operatorname{diam}\tilde{E} \ge \operatorname{diam} E$, (3.9) contradicts the monotonicity property of the function $A(d)$ in Lemma 1.

Assume that $l_n \neq \emptyset$. Then $f'(1) = f'(-1) = 0$. To prove that $|f'(e^{i\theta})| < \beta$ for all $e^{i\theta} \in l_n \setminus \{\pm 1\}$, we assume that $\beta = |f'(e^{i\theta_1})| < |f'(e^{i\theta_2})| = \beta_2$ with $e^{i\theta_1} \in l_f$ and some $e^{i\theta_2} \in l_n \setminus \{\pm 1\}$. Then applying the two point variation as above we get (3.6) and (3.9), again contradicting the monotonicity property of $A(d)$. Hence, $|f'(e^{i\theta})| \leq \beta$ for all $e^{i\theta} \in l_n$, which combined with the strict monotonicity property of Corollary 1 leads to the strict inequality in (3.2). □

Lemma 8 *If* $1 < d < 2$*, then* $l_n = \{e^{i\theta} : -\theta_0 < \theta < \theta_0\} \cup \{e^{i\theta} : \pi - \theta_0 < \theta < \pi + \theta_0\}$ *with some* $0 < \theta_0 = \theta_0(d) < \pi/2$*;* f' *is continuous on* $\overline{\mathbb{U}}^*$ *and for all* $z \in \overline{\mathbb{U}}^*$

$$|f'(z)| \leq |f'(e^{i\theta})| = \beta, \tag{3.10}$$

where $e^{i\theta} \in \bar{l}_f$ *and* $\beta > 1$.

Proof. Consider the function $g(z) = 1/f(1/z)$. By Lemma 2, g maps $\mathbb{U}$ onto a Jordan rectifiable domain possibly slit along two symmetric radial segments lying on the real axis. The double symmetry of $D = f(\mathbb{U}^*)$ implies that $G = g(\mathbb{U})$ is starlike w.r.t. $w = 0$. Since G is rectifiable and starlike, it follows from classical results of Lavrent'ev, see [P, p.163], that G is a Smirnov domain (non-Jordan in general). This shows that $\log|g'(z)| = \log|f'(1/z)| - 2\log|zf(1/z)|$, and therefore $\log|f'(1/z)|$, can be represented by the Poisson integral

$$\log|f'(1/z)| = \frac{1}{2\pi}\int_0^{2\pi} P(r, \theta - t)\log|f'(e^{-it})|\,dt \tag{3.11}$$

with boundary values defined a.e. on $\mathbb{T}$, see [P, p. 155]. Equation (3.11) along with (3.1) and (3.2) shows that $1 = |f'(\infty)| \leq \beta$ with equality only for the function $f(z) \equiv z$.

If $l_n = \emptyset$, then (3.11) and (3.1) show that $|f'| = \beta$ identically on $\mathbb{U}^*$. Therefore, $f(z) \equiv z$ contradicting the condition $d = (1/2)\operatorname{diam}\partial f(\mathbb{U}^*) > 1$. Hence, $l_n \neq \emptyset$. The latter implies that f is analytic in vicinities of the points $z = 1$ and $z = -1$ and $f'(z)$ has a simple zero at $z = 1$, $z = -1$. Consider the function $h(z) = \log|f'(1/z)/(z^2 - 1)|$, which can be represented by the Poisson integral

$$h(z) = \frac{1}{2\pi}\int_0^{2\pi} P(r, \theta - t)\log|f'(e^{it})/(e^{2it} - 1)|\,dt. \tag{3.12}$$

Equation (3.12) and the previous analysis show that h is a bounded harmonic function on $\mathbb{U}$. Let h_1 be a bounded harmonic function on $\mathbb{U}$ with boundary values $\log(\beta/|z^2-1|)$ on l_f and $h(z)$ on l_n. Then $h_1 - h$ has nontangential limit 0 a.e. on $\mathbb{T}$. Therefore, $h_1 - h \equiv 0$ in $\mathbb{U}$. Hence, $|f'| = \beta$ everywhere on l_f.

Since D possesses the double symmetry we need to show only that f' is continuous at $e^{i\theta_0}$. By the symmetry principle, f can be continued analytically through l_n and f' can be continued analytically through l_f. This implies that f can be considered as a function analytic in a slit disk $\Delta_0 = \mathbb{U}_\varepsilon(e^{i\theta_0}) \setminus [(1-\varepsilon)e^{i\theta_0}, e^{i\theta_0}]$ with $\varepsilon > 0$ small enough.

Using the Julia-Wolff lemma, see [P, Proposition 4.13], boundedness of f', and well-known properties of the angular derivatives, see [P, Propositions 4.7, 4.9], one can prove that f' has a finite limit $f'(e^{i\theta_0})$, $|f'(e^{i\theta_0})| = \beta$, along any path in $\overline{\mathbb{U}}^*$ ending at $e^{i\theta_0}$. The details of this proof are similar to the arguments in Lemma 13 in [BS]. $\square$

Lemma 9 *Let f be extremal in Σ_d for $1 < d < 2$ and let $\varphi(z) = zf'(z)$. Then φ maps $\mathbb{U}^*$ univalently onto the complement $\Omega(\beta,\psi) = \overline{\mathbb{C}} \setminus F(\beta,\psi)$ of the double anchor $F(\beta,\psi)$ defined in Theorem 2 where β is defined by* (1.7) *and $\psi = (1/2)\cos^{-1}(8\beta^{-1} - 8\beta^{-2} - 1)$.*

Proof. 1) Let $g(z) = f'(\sqrt{z})$. Since $g(\bar{z}) = \overline{g(z)}$, the symmetry principle implies that g is analytic in $\mathbb{U}^*$. We will show that g is univalent there.

By Corollary 1, $|g(e^{i\theta})| = |f'(e^{i\theta/2})|$ strictly increases from 0 to β as θ runs from 0 to $2\theta_0$. Since $\arg g(e^{i\theta}) = \arg f'(e^{i\theta/2}) = (\pi-\theta)/2$ strictly decreases from $\pi/2$ to $\varphi_0 = \pi/2 - \theta_0$ as θ runs from 0 to $2\theta_0$, it follows that g maps the arc $\{e^{i\theta} : 0 \le \theta \le 2\theta_0\}$ one-to-one onto an analytic Jordan arc δ_1 lying in the domain $U_\beta^+ = \{w \in \mathbb{U}_\beta : \Re w > 0, \Im w > 0\}$ and connecting the points $w = 0$ and $w = \beta e^{i\varphi_0} = f'(e^{i\theta_0})$.

Since $f(\mathbb{U}^*)$ is starlike w.r.t. $w = 0$,

$$\Re \frac{e^{i\theta} f'(e^{i\theta})}{f(e^{i\theta})} \ge 0$$

for $0 \le \theta \le 2\pi$. Since $f(\mathbb{U}^*)$ is symmetric w.r.t. the coordinate axes, the latter inequality shows that $-\pi \le \arg f'(e^{i\theta}) \le \pi - \theta_0$ for $0 \le \theta \le \theta_0$. This combined with (3.10) implies that g maps the arc $\{e^{i\theta} : 2\theta_0 \le \theta \le \pi\}$ one-to-one onto the circular arc $\delta_2 = \{\beta e^{i\varphi} : 0 \le \varphi \le \varphi_0\}$ such that $g(e^{2i\theta_0}) =$

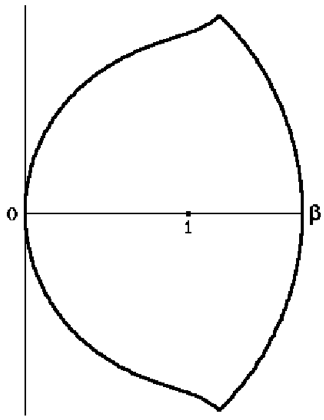

Figure 4: Domain G for $\beta = 1.7$

$\beta e^{i\varphi_0}$, $g(-1) = \beta$. By symmetry, g maps the arc $\{e^{i\theta} : -2\theta_0 \le \theta \le 0\}$ onto $\bar{\delta}_1 = \{w : \bar{w} \in \delta_1\}$ and the arc $\{e^{i\theta} : \pi \le \theta \le 2\pi - 2\theta_0\}$ onto $\bar{\delta}_2 = \{w : \bar{w} \in \delta_2\}$. Thus, g maps the unit circle $\mathbb{T}$ one-to-one onto a closed Jordan arc δ composed by δ_1, δ_2, $\bar{\delta}_2$, and $\bar{\delta}_1$. Since $g(\infty) = f'(\infty) = 1$ the argument principle implies that g maps $\mathbb{U}^*$ conformally and one-to-one onto a simply connected domain G which contains 1 and which is bounded by δ. The domain G for $\beta = 1.7$ is plotted in Figure 4.

The above mentioned properties of δ_1 show that G is circularly symmetric w.r.t. the positive real axis. Therefore by Lemma 4, $|g'(e^{i\theta})| = |f''(e^{i\theta/2})|$ strictly decreases in $2\theta_0 < \theta < \pi$.

2) Considering boundary values of φ we have

$$\Re\varphi(e^{i\theta}) = \Re e^{i\theta} f'(e^{i\theta}) = 0 \quad \text{for} \quad 0 \le \theta \le \theta_0$$

since $\Im f(e^{i\theta}) = 0$ for such θ. Since $\Im\varphi(e^{i\theta}) = |f'(e^{i\theta})|$ strictly increases in $0 \le \theta \le \theta_0$, φ maps l_n^{++} continuously and one-to-one onto the vertical segment $\{w : \Re w = 0, 0 \le \Im w \le \beta\}$.

For $\theta_0 \le \theta \le \pi/2$, $|\varphi(e^{i\theta})| = \beta$ and

$$\frac{\partial}{\partial\theta} \arg\varphi(e^{i\theta}) = \frac{\partial}{\partial\theta} \Im \log(e^{i\theta} f'(e^{i\theta})) = 1 + \frac{e^{i\theta} f''(e^{i\theta})}{f'(e^{i\theta})} = 1 - \beta^{-1}|f''(e^{i\theta})|,$$

since $e^{i\theta} f''(e^{i\theta})/f'(e^{i\theta})$ is real non-positive for $\theta_0 \le \theta \le \pi/2$.

Since $|f''(e^{i\theta})|$ strictly decreases in $\theta_0 \le \theta \le \pi/2$ it follows that $\frac{\partial}{\partial\theta} \arg\varphi(e^{i\theta})$ changes its sign at most once in the interval $\theta_0 < \theta < \pi/2$.

We have shown in 1) that $\arg f'(e^{i\theta})$ decreases from $\pi/2 - \theta_0$ to 0 when θ runs from θ_0 to $\pi/2$. Since $\arg\varphi(e^{i\theta_0}) = \pi/2$, the latter implies that

$$0 < \theta_0 < \arg\varphi(e^{i\theta}) = \theta + \arg f'(e^{i\theta}) < \pi - \theta_0 \tag{3.13}$$

for $\theta_0 < \theta < \pi/2$.

We claim that there is θ_1, $\theta_0 < \theta_1 < \pi/2$ such that

$$\begin{array}{lll} \frac{\partial}{\partial\theta} \arg\varphi(e^{i\theta}) < 0 & \text{if} & \theta_0 < \theta < \theta_1, \\ \frac{\partial}{\partial\theta} \arg\varphi(e^{i\theta}) > 0 & \text{if} & \theta_1 < \theta < \pi/2. \end{array} \tag{3.14}$$

Suppose to the contrary that $\frac{\partial}{\partial\theta} \arg\varphi(e^{i\theta}) \le 0$ for all $\theta_0 < \theta < \pi/2$. Then, we would have that $\arg\varphi(e^{i\theta})$ monotonically decreases over $\theta_0 < \theta < \pi/2$. Since $\varphi(e^{i\theta_0}) = \varphi(i) = i\beta$, we would have

$$\Delta \arg\varphi(e^{i\theta}) \Big|_{\theta_0}^{\pi/2} = -2\pi k \quad \text{for some} \quad k \in \mathbb{N}$$

contradicting (3.13). The assumption $\frac{\partial}{\partial\theta} \arg\varphi(e^{i\theta}) \ge 0$ for all $\theta_0 < \theta < \pi/2$ leads to the same contradiction. Since $|f''(e^{i\theta})|$ strictly decreases in $\theta_0 \le \theta \le \pi/2$, the claim follows.

Let $\psi = \arg\varphi(e^{i\theta_1})$. The previous analysis shows that $\theta_0 < \psi < \pi/2$ and φ maps each of the arcs $\{e^{i\theta} : \theta_0 \le \theta \le \theta_1\}$ and $\{e^{i\theta} : \theta_1 \le \theta \le \pi/2\}$ continuously and one-to-one onto the arc $\{\beta e^{it} : \psi \le t \le \pi/2\}$ such that $\varphi(e^{i\theta_1}) = \beta e^{i\psi}$. The symmetry principle now yields that φ maps the unit circle $\mathbb{T}$ continuously and one-to-one in the sense of boundary correspondence onto the boundary of the domain $\Omega(\beta,\psi)$. Hence by the argument principle, φ maps $\mathbb{U}^*$ conformally and univalently onto $\Omega(\beta,\psi)$

The normalization $f'(\infty) = 1$ leads after some work left to the interested readers to the relation $\psi = (1/2)\cos^{-1}(8\beta^{-1} - 8\beta^{-2} - 1)$. $\square$

4 Proofs of Theorems 1 and 2

To prove uniqueness of the extremal function in Σ_d, we assume that for some fixed d, $1 < d < 2$, there are distinct extremals f_1 and f_2. By Lemma 9,

$zf_k'(z) = g_{\beta_k}(z)$ for some $1 < \beta_1 < 2$, $1 < \beta_2 < 2$, where g_{β_k} maps $\mathbb{U}^*$ conformally onto the domain $\Omega(\beta_k) = \overline{\mathbb{C}} \setminus F(\beta_k, \psi(\beta_k))$. To be explicit, assume that $\beta_1 < \beta_2$. The domains $\Omega(\beta_1)$ and $\Omega(\beta_2)$ satisfy the conditions of Lemma 6. Therefore,

$$g_{\beta_1}(r) > g_{\beta_2}(r) \tag{4.1}$$

for all $r > 1$.

Since $f_k(\mathbb{U}^*)$ is doubly symmetric w.r.t. the real and imaginary axes, it follows that $a_0(f_k) = 0$ for $k = 1, 2$. Hence, it follows from the normalization in (1.5) that

$$\lim_{R\to\infty} (f_1(R) - f_2(R)) = 0. \tag{4.2}$$

On the other hand, since $f_1(1) = f_2(1) = d$, we have

$$f_1(R) - f_2(R) = \int_1^R t^{-1}(g_{\beta_1}(t) - g_{\beta_1}(t))\, dt$$

and (4.1) implies that the integrand is positive and hence, that $f_1(R)-f_2(R)$ is a (positive) increasing function of R, which contradicts (4.2).

Let f_d denote the unique extremal function in Σ_d. To find f_d explicitly, we represent $\varphi(z; d) = zf_d'(z)$ as

$$\varphi(z; d) = (F_{d^{-2}}(z^{-2}))^{-1/2},$$

where $F_p(\zeta) = \zeta + a_2(F_p)\zeta^2 + \ldots$, $1/4 \le p \le 1$, is the univalent function in the standard class S that maps the unit disk $\mathbb{U}$ onto the domain $\overline{\mathbb{C}} \setminus ((-\infty, -p] \cup \{pe^{i\tau} : |\tau - \pi| \le \alpha\})$ with $\alpha = \cos^{-1}(8\sqrt{p} - 8p - 1)$. It is well known that F_p is extremal in a number of problems, for instance in the problem on $\max|a_2(f)|$ studied by E. Netanyahu [N] and T. Suffridge [S], on the subclass of functions $f \in S$ that cover the disk $\mathbb{U}_p$. Using an explicit expression for F_p, see for example, [S], we get (1.9) and after an integration (1.7).

The integral in (1.8) can be evaluated in terms of elementary functions. We leave to the interested readers to check (one can use "Mathematica" or "Maple") that $f_d(1)$ coincides with the right-hand side in (1.7), which in this case is equivalent to the equality $f_d(1) = d$. Since for each $1 \le d \le 2$ the extremal function is unique in Σ_d, (1.7) has a unique solution $\beta = \beta(d)$ on the interval $1 \le \beta \le 2$; this also follows easily from the monotonicity of the right-hand side of (1.7).

To evaluate the maximal omitted area $A(d) = \mathrm{Area}\,(E_{f_d})$, we apply a standard line integral formula and the fact that $\Im\,(\bar{w}\,dw) = 0$ on the non-free boundary. We have

$$A(d) = \frac{1}{2}\Im \int_{\partial E_{f_d}} \bar{w}\,dw = \frac{1}{2}\Im \int_{L_{f_d}} \bar{w}\,dw = \frac{1}{2}\Re \int_{l_{f_d}} \overline{f_d(e^{i\theta})} e^{i\theta} f_d'(e^{i\theta})\,d\theta.$$

Since $|f_d'|^2 = \beta^2$ on l_{f_d}, we obtain

$$A(d) = \frac{\beta^2}{2}\Re \lim_{\varepsilon\to+0}\left\{\int_{\varepsilon}^{\pi-\varepsilon} \frac{f_d(e^{i\theta})}{e^{i\theta} f_d'(e^{i\theta})}\,d\theta + \int_{\pi+\varepsilon}^{2\pi-\varepsilon} \frac{f_d(e^{i\theta})}{e^{i\theta} f_d'(e^{i\theta})}\,d\theta\right\} =$$

$$= \frac{\beta^2}{2}\Im\left\{\int_{\mathbb{T}} \frac{f_d(z)}{z^2 f_d'(z)}\,dz - \pi i \mathrm{Res}\left[\frac{f_d(z)}{z^2 f_d'(z)}, 1\right] - \pi i \mathrm{Res}\left[\frac{f_d(z)}{z^2 f_d'(z)}, -1\right]\right\},$$

where $\int_{\mathbb{T}} f_d/(z^2 f_d')\,dz$ is understood as the Cauchy principal value. The function $f_d/(z^2 f_d')$ has simple poles at $z=1$ and $z=-1$. Computing the integral and residues, we obtain

$$A(d) = \pi[\beta^2 - 2d(\beta-1)],$$

which implies (1.6). This finishes the proof of Theorem 2.

To deduce (1.4), we write (1.6) in an invariant form:

$$\frac{\mathrm{area}\,E}{(\pi/4)\mathrm{diam}^2 E} \le p^{-2}[\beta^2(p) - 2p(\beta(p)-1)] \tag{4.3}$$

with $p = \mathrm{diam}\,E/(2\mathrm{cap}\,E)$, where $1 \le \beta(p) \le 2$ is defined by (1.7) with d replaced by p. Since the maximal omitted area $A(d)$ strictly decreases, the expression in the brackets in (4.3) decreases and therefore the right-hand side of (4.3) itself decreases from 1 to 0 when p runs from 1 to 2. Therefore there is a function $p = \Psi_1(s)$ inverse to $s = p^{-2}[\beta(p)^2 - 2p(\beta(p)-1)]$. Let $\Psi(s) = 1/\Psi_1(s)$. Since the inverse Ψ_1 is decreasing, (4.3) leads to the inequality

$$p \le \Psi_1(\mathrm{area}\,E/((\pi/4)\mathrm{diam}^2 E),$$

which is equivalent to (1.4) with equality only for the continua described in Theorem 1.

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