Proceedings of the International Conference on Geometric Function Theory,
Special Functions and Applications (ICGFT)

The Verification of an Inequality

Roger W. Barnard and Kent Pearce

Abstract. In a recent paper, we verifed a conjecture of Mejía and Pommerenke that the extremal value for the Schwarzian derivative of a hyperbolically convex function is realized by a symmetric hyperbolic "strip" mapping. There were three major steps in the verification: first, a variational argument was given to reduce the problem to hyperbolic polygons bounded by at most two hyperbolic geodesics; second, a reduction was made to hyperbolic polygons bounded by exactly two symmetric hyperbolic geodesics; third, for hyperbolic polygons bounded by exactly two symmetric hyperbolic geodesics a computation was made, using properties of special functions, to find the maximal value of the Schwarzian derviative.

In between the second and third steps, an assertion was made that "using an extensive computational argument which considers several cases" the problem of computing the Schwarzian derivative for hyperbolic polygons bounded by exactly two symmetric hyperbolic geodesics could be reduced to computing the Schwarzian derivative for hyperbolic polygons bounded by exactly two symmetric hyperbolic geodesics under the assumption that the argument z of the Schwarzian derviative satisfied the restriction $0 \le z < 1$. In this paper, we provide a verification for that assertion.

Keywords. Hyperbolically Convex, Schwarzian Derivative.

2000 MSC. 30C70.

1. Introduction

Hyperbolic convexity is a natural generalization of euclidean convexity; a region G in the Poincaré model $\mathbb{D}$ of the hyperbolic plane is **hyperbolically convex** if for any two points in G, the hyperbolic geodesic segment between them lies entirely in G. Such regions arise naturally in Teichmüller theory, for example, since the fundamental domains of Fuchsian groups are hyperbolically convex [3, 5].

A conformal map $f : \mathbb{D} \to \mathbb{D}$ is hyperbolically convex if its range is hyperbolically convex. Hyperbolically convex functions have been extensively studied by Ma and Minda [6, 7] and Mejía and Pommerenke [8–12], as well as Beardon [3] and Solynin [16, 17], among others. One frequently cited open problem was to find for hyperbolically convex functions a sharp bound on the Schwarz norm

$$||S_f||_{\mathbb{D}} = \sup\{|S_f(z)|\eta_{\mathbb{D}}^{-2}(z) : z \in \mathbb{D}\},$$

Version April 4, 2008.
The authors were partially supported by the National Science Foundation grant DMS-03040000.

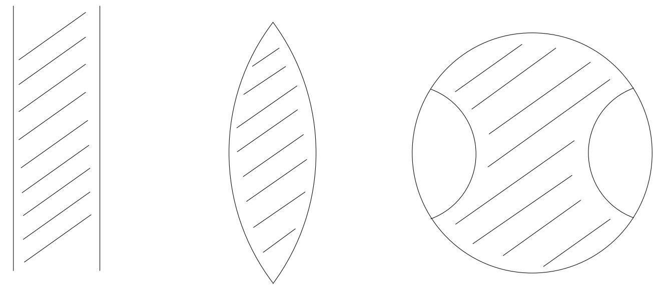

FIGURE 1. The extremal domains for maximizing the Schwarz norm in euclidean (left), spherical (center), and hyperbolic (right) geometry.

where $\eta_{\mathbb{D}}(z) = \frac{1}{1-|z|^2}$ is the hyperbolic density of $\mathbb{D}$ and S_f is the Schwarzian derivative

$$S_f = \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2.$$

The Schwarz norm of an analytic function f has long been a primary tool in understanding its geometric behavior. For example, $||S_f||_{\mathbb{D}} = 0$ if and only if f is a Möbius transformation. Thus $||S_f||_{\mathbb{D}}$ is thought of as measuring how closely the geometric behavior of f resembles that of a Möbius transformation. Since the image of $\mathbb{D}$ under a Möbius transformation must be a disc, $||S_f||_{\mathbb{D}}$ also measures the difference between the conformal geometry of $f(\mathbb{D})$ and that of a disc. Lehto has used this idea to great effect, producing a pseudo-metric on the set of all simply connected proper subdomains of $\mathbb{C}$. See Lehto's book [5], for example, for a detailed discussion.

Nehari showed that if $f(\mathbb{D})$ is convex (in the euclidean sense), then $||S_f||_{\mathbb{D}} \leq 2$, with equality if and only if $f(\mathbb{D})$ is an "infinite strip" bounded by two parallel lines [15]. Similarly, Mejía and Pommerenke showed that the extremal domain for spherically convex functions is a "spherical strip" [9].

The problem of finding a similar bound for the Schwarz norm of hyperbolically convex functions has been intensely studied by a number of authors, including Ma, Minda, Mejía, Pommerenke and Vasilev [6–8, 11–13]. Mejía and Pommerenke [8] found partial results on the bound and conjectured that the extremal value of $||S_f||_{\mathbb{D}}$ is attained by a map of the form

$$f_\alpha(z) = \tan\left(\alpha \int_0^z \left(1 - 2\xi^2\cos 2\theta + \xi^4\right)^{-1/2} d\xi\right), \tag{1.1}$$

where $\alpha = \frac{\pi}{2K(\cos\theta)}$, and K is the elliptic integral of the first kind. The range of f_α is a "hyperbolic strip" bounded by two geodesics through $\pm\tanh\left(\frac{\pi K(\sin\theta)}{4K(\cos\theta)}\right)$ and perpendicular to the real axis. See Figure 1.

In [2], we verified Mejía and Pommerenke's conjecture and completed the classification of the extremal domains for the Schwarzian in all three of the classical geometries. Specifically, we proved

Theorem 1.1. *The maximal value of the Schwarz norm for hyperbolically convex functions is* $S_{f_\alpha}(0)$*, where*

$$f_\alpha(z) = \tan\left(\alpha \int_0^z \left(1 - 2\xi^2 \cos 2\theta + \xi^4\right)^{-1/2} d\xi\right), \qquad \alpha = \frac{\pi}{2K(\cos\theta)},$$

K *is the elliptic integral of the first kind, and* α *is chosen so that* $\cos\theta$ *is the unique critical point of the function*

$$g(s) = 4s^2 - 2 + \frac{\pi^2}{2K^2(s)} \tag{1.2}$$

on $(0, 1)$.

A computer calculation produces a maximal value for the Schwarz norm for a hyperbolically convex function of approximately 2.383635.

The proof of Theorem 1.1 in [2] can be summarized as follows. The first step described a class of hyperbolic polygons which is dense in the class of all hyperbolically convex functions, to which the problem of computing the Schwarz norm could be reduced. The second step developed, using the Julia variational formula, two class preserving variations for the described class of hyperbolic polygons. The third step used the first of the developed variations to reduce the problem to computing the Schwarz norm for hyperbolic polygons bounded by at most four "proper" hyperbolic geodesics. The fourth step used the second of the developed variations to reduce the problem to computing the Schwarz norm for hyperbolic polygons bounded by at most two "proper" hyperbolic geodesics. The fifth step showed by explicit calculations that hyperbolic polygons bounded by exactly one side or by two intersecting sides could not be extremal, i.e., the fifth step reduced the problem to computing the Schwar norm for hyperbolic polygons of the form f_α as given in equation (1.1). The sixth step gave an argument, using properties of special functions, which showed that among the functions described by equation (1.1) a unique extremal exists.

In between steps five and six, an assertion in [2] was made that "using an extensive computational argument which considers several cases" the problem of computing the Schwarzian derivative for hyperbolic polygons of the form f_α as given in equation (1.1) could be reduced to computing the Schwarzian derivative for hyperbolic polygons of the form f_α as given in equation (1.1) under the assumption that the argument z of the Schwarzian derviative satisfied the restriction $0 \leq z < 1$. In this paper, we provide the details for the verification of that assertion.

In Section 2 we develop background material on hyperbolic convexity and the Schwarzian derivative and discuss more of the history of the problem. In Section 3 we provide the technical details for the verification of the assertion described above.

2. Hyperbolic Convexity and Schwarzians

2.1. Hyperbolic Geometry. The unit disc $\mathbb{D}$ equipped with the metric

$$d_h(z,w) = \inf\left\{\int_\gamma \frac{1}{1-|z|^2}\,|dz| : \gamma \text{ is a rectifiable curve joining } z \text{ and } w\right\}$$

forms a model for the hyperbolic plane [3]. Notice that the Poincaré density

$$\eta_{\mathbb{D}}(z) = \frac{1}{1-|z|^2}$$

goes to infinity as z moves toward the boundary of the disc. Consequently, integrating $\eta_{\mathbb{D}}$ over curves near the boundary produces large values of the integral. If z and w do not lie on a ray through the origin, then the euclidean line segment joining them will produce a larger integral than a curve which bends away from the boundary. In fact, the infimum will be achieved by an arc of a circle perpendicular to $\partial\mathbb{D}$. Such curves are hyperbolic geodesics. Since disc automorphisms

$$M(z) = e^{i\theta}\frac{z-a}{1-\overline{a}z},$$

where $\theta \in [0, 2\pi)$ and $a \in \mathbb{D}$, preserve circles orthogonal to $\partial\mathbb{D}$, they are precisely the isometries of $\mathbb{D}$.

Any region G conformally equivalent to $\mathbb{D}$ also carries a hyperbolic metric defined in the same manner using the density

$$\eta_G(z) = \frac{|f'(z)|}{1-|f(z)|^2},$$

where f is a conformal map of G onto $\mathbb{D}$. Notice that it doesn't matter which map f is chosen as any two such maps must differ by a disc automorphism.

2.2. Convexity. The euclidean notion of convexity generalizes to hyperbolic regions in an obvious manner.

Definition 2.1. A region $\Omega \subset \mathbb{D}$ is **hyperbolically convex** if for any two points $z, w \in \Omega$, the hyperbolic geodesic segment joining z and w lies completely in Ω.

Notice that since the disc automorphisms are the isometries of the hyperbolic plane, the image $M(\Omega)$ of Ω under a disc automorphism M is hyperbolically convex if and only if Ω is hyperbolically convex. The fundamental domains of discrete groups of disc automorphisms provide a great many useful examples of hyperbolically convex domains. See Beardon [3] for an extensive discussion of these regions.

We will call a hyperbolically convex region Ω bounded by a finite number of either geodesic arcs lying inside $\mathbb{D}$ or arcs of $\partial\mathbb{D}$ a **hyperbolically convex polygon**. We call the bounding geodesic arcs **proper sides** and the arcs of $\partial\mathbb{D}$ **improper sides**. For $n \geq 0$, we let

$$\begin{aligned} K_n = \{&\text{hyperbolically convex polygons containing } 0 \\ &\text{and having at most } n \text{ proper sides}\} \cup \{0\}. \end{aligned}$$

Definition 2.2. A conformal map $f : \mathbb{D} \to \Omega$ is called a **hyperbolically convex function** if its range is hyperbolically convex. We let H denote the class of all hyperbolically convex functions that fix the origin and let H_n denote the subset of functions whose range is in K_n.

2.3. Schwarzians. Much of the geometric behavior of an analytic function is described by its Schwarzian derivative [4, 5].

Definition 2.3. The **Schwarzian derivative** (or just "Schwarzian") of an analytic function f is

$$S_f = \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2.$$

Proposition 2.4. *The Schwarzian of an analytic function is identically* 0 *if and only if it is a Möbius transformation. Moreover, the Schwarzian satisfies the chain rule*

$$S_{f\circ g} = (S_f \circ g)\,(g')^2 + S_g.$$

Thus, if M is Möbius, then

$$S_{M\circ g} = S_g.$$

and

$$S_{f\circ M} = (S_f \circ M)\,(M')^2.$$

Hence the Schwarzian is unchanged by post-composition with a Möbius transformation, but pre-composition produces an extra quadratic factor.

Definition 2.5. Let f be defined on a simply connected region $G \subsetneq \mathbb{C}$. The **Schwarz norm** of f is given by

$$||S_f||_G = \sup_{z\in G} \eta_G^{-2}(z)|S_f(z)|.$$

By taking into account the density of the hyperbolic metric, the Schwarz norm is completely Möbius invariant. It is easy to show for any Möbius M that

$$\eta_{M^{-1}(G)}(z) = \eta_G(M(z))|M'(z)|,$$

and thus

$$\frac{|S_{f\circ M}(z)|}{\eta^2_{M^{-1}(G)}(z)} = \frac{|S_f(M(z))|\,|M'(z)|^2}{\eta_G^2(M(z))\,|M'(z)|^2} = \frac{|S_f(w)|}{\eta_G^2(w)}.$$

where $w = M(z)$. Thus

$$||S_f||_G = ||S_{f\circ M}||_{M^{-1}(G)}$$

and

$$||S_f||_G = ||S_{M\circ f}||_G.$$

In particular, notice that $||S_f||_{\mathbb{D}}$ is unchanged by disc automorphisms.

2.4. Geometry of the Schwarzian. Since $||S_f||_{\mathbb{D}} = 0$ if and only if f is Möbius, we can view $||S_f||_{\mathbb{D}}$ as measuring how close f is to being a Möbius transformation. Since any Möbius transformation would send $\mathbb{D}$ to another disc or half plane, $||S_f||_{\mathbb{D}}$ also measures the amount of deformation between $f(\mathbb{D})$ and a disc. This notion was formalized by Lehto [5] to produce a pseudometric between regions conformally equivalent to a disc.

There are a number of results that show that if $||S_f||_{\mathbb{D}}$ is small, then $f(\mathbb{D})$ possesses disc-like properties. The two most important for our purposes are due to Nehari [14, 15].

Theorem 2.6. *If $||S_f||_{\mathbb{D}} < 2$, then f is univalent and $f(\mathbb{D})$ is a quasidisc. Moreover, if f is univalent, then $||S_f||_{\mathbb{D}} \le 6$.*

Theorem 2.7. *If $f(\mathbb{D})$ is convex (in the euclidean sense), then $||S_f||_{\mathbb{D}} \le 2$, with equality if and only if $f(\mathbb{D})$ is an infinite strip.*

Mejía and Pommerenke [9] proved a similar result for spherically convex regions.

Theorem 2.8. *If $f(\mathbb{D})$ is spherically convex, then*

$$||S_f||_{\mathbb{D}} \le 2(1 - \sigma(f)^2),$$

where

$$\sigma(f) = \max_{z \in \mathbb{D}} (1 - |z|^2) \frac{|f'|}{1 + |f|^2}.$$

For a fixed value of $\sigma(f)$, this maximum value of $||S_f||_{\mathbb{D}}$ is achieved by a map of the form $f_\phi(z) = i \tanh\left(\frac{2\phi}{\pi} \operatorname{Log}\left(\frac{1+z}{1-z}\right)\right)$ which takes $\mathbb{D}$ onto a "spherical strip," that is, a lune bounded by great circles through $\pm i$ and making an angle 2ϕ with the imaginary axis.

Thus, convex and spherically convex regions cannot be deformed too far from being a disc in the sense of the Lehto pseudometric, and the regions with the greatest amount of deformation are strips. It was been conjectured by Mejía and Pommerenke [8,10,12] that the same must hold for hyperbolically convex regions. Theorem 1.1 in [2] verified this conjecture for hyperbolically convex regions.

3. Verification

3.1. Preliminaries. A direct computation shows that the Schwarzian derivative of f_α given in (1.1) is

$$S_{f_\alpha}(z) = 2(c + \alpha^2) \frac{1 - 2dz^2 + z^4}{(1 - 2cz^2 + z^4)^2},$$

where

$$c = \cos(2\theta), \quad \alpha = \frac{\pi}{2K(\cos\theta)}, \quad d = \frac{3 + 2\alpha^2 c - c^2}{2(c + \alpha^2)}.$$

We note that the parameters $c, \ \alpha, \ d$ are functions of an underlying auxillary parameter θ. We will write $\alpha(\theta)$ if we need to emphasize the dependence of α on θ.

We also note, using symmetry, that the computation of the Schwarz norm

$$||S_{f_\alpha}||_{\mathbb{D}} = \sup_{z \in \mathbb{D}} (1-|z|^2)^2 |S_{f_\alpha}(z)|$$

is equivalent to finding

$$\sup_{z \in \mathbb{D}^+} (1-|z|)^2 2(c+\alpha^2) \left| \frac{1-2dz+z^2}{(1-2cz+z^2)^2} \right| \tag{3.1}$$

where $\mathbb{D}^+ = \{z = re^{i\phi} : 0 \le r < 1, \ \ 0 \le \phi \le \pi\}$.

The assertion which we need to verify is

Assertion 1. *The problem of maximizing*

$$(1-|z|)^2 2(c+\alpha^2) \frac{1-2dz+z^2}{(1-2cz+z^2)^2}$$

over $\mathbb{D}^+$ *over all values of* $\theta, \ 0 < \theta < \pi/2$, *can be reduced to the problem of maximizing*

$$(1-x)^2 2(c+\alpha^2) \frac{1-2dx+x^2}{(1-2cx+x^2)^2}$$

over $0 \le x < 1$ *over all values of* $\theta, \ 0 < \theta < \pi/2$.

We note, as an aside, that Theorem 1.1 showed that for this latter problem the maximum value is greater that 2.

We recall for the reader that in the proof of Theorem 1.1 in [2] it was shown that the factor $c + \alpha^2$ in (3.1) satisfies the following conditions:

a. $c + \alpha^2 > 0$ for $0 < \theta < \pi/2$,
b. $c + \alpha^2|_{\theta=0} = 1$, $c + \alpha^2|_{\theta=\pi/2} = 0$,
c. $c + \alpha^2$ is unimodal on $0 < \theta < \pi/2$, i.e. there exists a unique θ_* in $(0, \pi/2)$ such $c + \alpha^2$ is strictly increasing on $(0, \theta_*)$ and strictly decreasing on $(\theta_*, \pi/2)$.

The value θ_* is the unique value so that $s = \cos\theta_*$ is the unique critcial point of equation (1.2). Numerically, $\theta_* \approx 0.218$.

Consequently, there exists a unique $\theta_0, \ \theta_* < \theta_0 < \pi/2$ such that:

d. $1 < c + \alpha^2$ for $0 < \theta < \theta_0$,
e. $0 < c + \alpha^2 < 1$ for $\theta_0 < \theta < \pi/2$.

Numerically, $\theta_0 \approx 0.554$. See Figure 2.

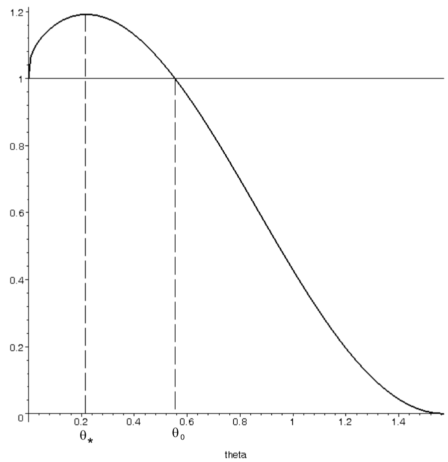

FIGURE 2. The curve $c + \alpha^2$ and the parameters θ_* and θ_0.

3.2. Lemmas. In order to verify Assertion 1, we will need analytic estimates on the parameter α which is defined in terms of the elliptic integral K. We recall the following facts about the behavior of elliptic integrals [1, pp. 53-54]

Lemma 3.1. *Let K be the complete elliptic integral of the first kind. Then,*

1. *the function $K(y)/\log(e^2/\sqrt{1-y^2}\,)$ is strictly decreasing from $(0,1)$ onto $(\pi/4, 1)$,*
2. *for each $c \in [1/2, \infty)$ the function $(\sqrt{1-y^2})^c\, K(y)$ is decreasing from $(0,1)$ onto $(0, \pi/2]$.*

As a consequence of Lemma 3.1, we have the following upper and lower bounds for the parameter $\alpha = \alpha(y)$, where $y = \cos(\theta)$

Lemma 3.2. *Let $\alpha = \dfrac{\pi/2}{K(y)}$. Then,*

1. $\alpha \leq \dfrac{1}{1 - \frac{\log(1-y^2)}{4}}$, *for $0 < y < 1$,*
2. *for each $c \in [1/2, \infty)$, $(\sqrt{1-y^2})^c < \alpha$ for $0 < y < 1$.*

We note that if the upper bound for α in Lemma 3.2 is expanded as a MacLaurin series in y, then all of the coefficients in the series expansion, except for the constant term (which is 1), are negative. Hence, any partial sum of the series expansion

$$\frac{1}{1 - \frac{\log(1-y^2)}{4}} = 1 - \frac{1}{4}y^2 - \frac{1}{16}y^4 - \frac{7}{192}y^6 - \frac{19}{768}y^8 - \cdots \tag{3.2}$$

will also provide an upper bound for α.

Finally, to estimate lower bounds for polynomials in two variables with rational coefficients we will need the following technical lemma.

Lemma 3.3. *Let $p = p(x,y)$ be a polynomial with rational coefficients defined on the region $R = \{(x,y) : a \le x \le b, \quad c \le y \le d\}$, where a, b, c, d are all rational numbers. Let $M_x \ge \max_{(x,y)\in R} \left|\frac{\partial p}{\partial x}(x,y)\right|$, $M_y \ge max_{(x,y)\in R} \left|\frac{\partial p}{\partial y}(x,y)\right|$ and $M = \max\{M_x, M_y\}$. Let $\delta > 0$ and suppose N_w, N_h are chosen so that $\Delta_w = (b-a)/N_w \le \delta$, $\Delta_h = (d-c)/N_h \le \delta$. Let L be the lattice $L = L(N_w, N_h) = \{(a + i\Delta_w, c + j\Delta_h) \ : \ 0 \le i \le N_w, \quad 0 \le j \le N_h\}$. Let $m = min_{(x,y)\in L} p(x,y)$. If $m \ge M\delta$, then $p(x,y) \ge 0$ on R.*

Proof. Let $(x,y) \in R$. Then there exists $(x_0,y_0) \in L$ such $\mathrm{dist}((x_0,y_0),(x,y)) \le \delta/\sqrt{2}$. We can write $(x,y) = (x_0,y_0) + s(\cos\tau, \sin\tau)$ where $0 < s \le \delta/\sqrt{2}$. Define $\tilde{p}(t) = p((x_0,y_0) + t(\cos\tau,\sin\tau))$ where $0 \le t \le s$. Then, $|\tilde{p}'(t)| \le M\sqrt{2}$ for $0 \le t \le s$. Hence, $p(x,y) = \tilde{p}(t) \ge \tilde{p}(0) - M\sqrt{2}\,t = p(x_0,y_0) - M\sqrt{2}\,t \ge m - M\delta \ge 0$. ■

To verify Assertion 1, we will consider the following cases for the parameters ϕ, θ and r in (3.1)

- Case 1. $0 < \phi < \pi$, $0 < r < 1$
  - Case 1a. $0 < \theta < \theta_0$
  - Case 1b. $\theta_0 < \theta < \pi/2$
- Case 2. $\phi = \pi$, $0 < \theta < \pi$, $0 < r < 1$
- Case 3. $\phi = 0$, $0 < \theta < \pi$, $0 \le r < 1$

We note that Case 3 is the conclusion of Assertion 1, i.e., Case 3 is the case which was explicitly detailed in [2]. Hence, we will address the other cases here, i.e., we will show that for $0 < r < 1$ and for ϕ and θ restricted to either Case 1 or Case 2, then (3.1) is not maximizied.

Let $h(r,\phi)$ denote the function in (3.1), i.e.,

$$h(r,\phi) = (1 - |z|)^2 2(c + \alpha^2)\left|\frac{1 - 2dz + z^2}{(1 - 2cz + z^2)^2}\right|$$

Before we begin the verification, we will establish the following lemma which gives bounds the location of the critical points of $h(r,\phi)$.

Lemma 3.4. *If $0 < \phi < \pi$, then for each fixed θ, $\quad 0 < \theta < \pi/2$, the function $h(r,\phi)$ has a unique critical point (r_θ, ϕ_θ) in $\mathbb{D}^+$ given by*

$$\cos\phi_\theta = \frac{c + \sqrt{c^2+3} - d}{2} \tag{3.3}$$

$$r_\theta^2 + (1 - d(c + \sqrt{c^2+3})r_\theta + 1 = 0 \tag{3.4}$$

Furthermore, for $r_0 = \min_{0<\theta<\pi/2} r_\theta$ we have $r_0 \ge 2/5$.

Proof. Let $H(r,\phi) = (1-r)^4 X\bar{X}$, where $X = \dfrac{1-2dz+z^2}{(1-2cz+z^2)^2}$. Solving simultaneously the system of equations

$$\frac{\partial H}{\partial r} = 0$$
$$\frac{\partial H}{\partial \phi} = 0$$

for the critical points of H yields the conditions that

$$\begin{aligned} -4X\bar{X} \quad &+ \quad (1-r)X'\bar{X}\frac{z}{r} \; + \; (1-r)X\bar{X}'\frac{\bar{z}}{r} = 0 \\ X'\bar{X}iz \quad &+ \quad X\bar{X}'(-i\bar{z}) = 0. \end{aligned}$$

where X' denotes $\dfrac{\partial X}{\partial z}$. Consequently, we have

$$\frac{2r}{1-r} = \frac{zX'}{X} = \frac{-2dz+2z^2}{1-2dz+z^2} - 2\frac{-2cz+2z^2}{1-2cz+z^2}$$

which implies

$$\frac{1+r}{1-r} = \frac{1-2(2d-c)z+z^2}{1-2dz+z^2}\frac{1-z^2}{1-2cz+z^2}. \tag{3.5}$$

If we multiply equation (3.5) by $(1-r)(1-2dz+z^2)(1-2cz+z^2)$, collect terms and then set imaginary and real parts equal to zero we obtain

$$\begin{aligned} (1-r)\sin\phi[(d-2c+2\cos\phi)r^2 - 2(d+c-\cos\phi)r \\ +(d-2c+2\cos\phi)] &= 0 \\ (1+r)\{[(d-2c+2\cos\phi)\cos\phi - 1]r^2 \\ -2[(2d-c+\cos\phi)\cos\phi - cd - 1]r + [(d-2c+2\cos\phi)\cos\phi - 1]\} &= 0 \end{aligned} \tag{3.6}$$

Since $\sin\phi \neq 0$, then we have, solving these equations simultaneously

$$2(c-\cos\phi)(4\cos^2\phi - 4(c-d)\cos\phi + d^2 - 2cd - 3) = 0$$

Substituting the factor $\cos\phi = c$ into (3.6) forces $r = 1$. There is a unique solution of the factor $(4\cos^2\phi - 4(c-d)\cos\phi + d^2 - 2cd - 3) = 0$ for $0 < \phi < \pi$, namely the solution given in (3.3). Substituting this solution into (3.6) yields the quadratic constraint in (3.3) on r. Hence, we have, given that $\sin\phi \neq 0$, then there is a unique critical point (r_θ, ϕ_θ) in $\mathbb{D}^+$ which satisfies (3.3).

To show that $r_0 \geq 2/5$, we will show that each r_θ satisfies the inequality $2/5 < r_\theta < 1$. The latter is equivalent to $3 < d(c+\sqrt{c^2+3}) < 49/10$.

We note that the inequality $2-c < d$ would imply that

$$(2-c)(c+\sqrt{c^2+3}) < d(c+\sqrt{c^2+3}).$$

It is straightforward to verify that $3 < (2-c)(c+\sqrt{c^2+3})$. Also, the inequality $d < 2-c+1/2$ would imply that

$$d(c+\sqrt{c^2+3}) < (2-c+1/2)(c+\sqrt{c^2+3}).$$

It is straightforward to verify that $(2-c+1/2)(c+\sqrt{c^2+3}) < 49/10$.

Thus, it remains to verify that $2-c<d$ and $d<2-c+1/2$. First, we note that

$$d-(2-c)=\frac{3+4\alpha^2c+c^2-c-4\alpha^2}{2(c+\alpha^2)}=\frac{q_1}{2(c+\alpha^2)}.$$

Making a change of variable $c=2y^2-1$ (where $y=\cos\theta$), we have

$$q_1=(8y^2-8)\alpha^2+8+4y^4-12y^2$$

Note that the coefficient of α^2 in q_1 is negative. Since we are looking for a lower bound for q_1, we replace $\alpha=\alpha(y)$ by an upper bound obtained from Lemma 3.2. Specifically, we bound α by the 2^{nd}-order partial sum

$$\alpha \le p_2 = 1-\frac{1}{4}y^2.$$

We have then,

$$q_1 \ge q_1^* = q_1|_{\alpha=p_2} = \frac{y^4(1-y^2)(16-8y^2-y^4)}{32} > 0$$

Second, we have

$$(2-c+1/2)-d=\frac{5c+5\alpha^2-c^2-4\alpha^2c-3}{2(c+\alpha^2)}=\frac{q_2}{2(c+\alpha^2)}.$$

Making a change of variable $c=2y^2-1$ (where $y=\cos\theta$), we have

$$q_2=(9-8y^2)\alpha^2+14y^2-9-4y^4$$

Note that the coefficient of α^2 in q_2 is positive. Since we are looking for a lower bound for q_2, we replace $\alpha=\alpha(y)$ by a lower bound obtained from Lemma 3.2, with $c=1/2$. Specifically, we bound α by

$$\alpha \ge p_{1/2}=(1-y^2)^{1/4}.$$

We have then,

$$q_2 \ge q_2^* = q_2|_{\alpha=p_{1/2})} = (9-8y^2)\sqrt{1-y^2}+14y^2-9-4y^4$$

To show that $q_2^*>0$, one isolates the square root, squares the terms in the resulting inequality and then collects all of the terms to obtain a new inequality of the form

$$p(y)=-16y^8+48y^6-60y^4+27y^2>0$$

A Sturm sequence argument then verifies $p(y)>0$ on the interval $0<y<19/20$. On the other hand, clearly $q_2^*(y)>14y^2-9-4y^4$ and the latter quartic is non-negative on $19/20<y<1$. ∎

3.3. Verification of Assertion 1. Case 1a. $0 < \phi < \pi, \quad 0 < \theta < \theta_0, \quad 0 < r < 1$

We will show that $S_{f_\alpha}(0) > h(r,\phi)$. Let

$$(S_{f_\alpha}(0))^2 - |h(r,\phi)|^2 = \frac{p_1(\theta,r,x)}{q_1(\theta,r,x)}$$

where $x = \cos\phi$. It is easily seen that the numerator p_1 is a reflexive 6^{th}-degree polynomial in r. Making a change of variable $r = e^{-s}$, we can write

$$e^{3s} p_1(\theta,r,x) = p_2(\theta,\cosh s,x)$$

where p_2 is a 3^{rd}-degree polynomial in $\cosh s$. We substitute $\cosh s = 1 + 2\sinh^2(s/2)$ into p_2 to obtain

$$p_3(\theta,\sinh(s/2),x) = p_2(\theta,1+2\sinh^2(s/2),x)$$

which is an even 6^{th}-degree polynomial in $\sinh(s/2)$. Finally, we make a change of variable $\sqrt{t} = \sinh(s/2)$ to obtain

$$p_4(\theta,t,x) = p_3(\theta,\sinh(s/2),x)$$

which is a 3^{rd}-degree polynomial in t.

We have reduced our problem to showing that

$$p_4(t) = p_4(\theta,t,x) = c_3(\theta,x)t^3 + c_2(\theta,x)t^2 + c_1(\theta,x)t + c_0(\theta,x) > 0$$

for $t > 0$ under the assumption that $0 < \theta < \theta_0$. It suffices to show that p_4 as a function of t is totally monontonic, i.e., to show that each coefficient $c_j = c_j(\theta,x) \geq 0, \quad j = 0 \cdots 3$, where

$$\begin{aligned} c_3 &= 16[(d-2c)x+1] \\ c_2 &= 4[(1+4c^2)x^2 + (-12c+2d)x + 2c^2 - d^2] \\ c_1 &= 8[(1-cx)(x-c)^2] \\ c_0 &= (x-c)^4 \end{aligned}$$

Since c_3 is linear in x, we have that

$$c_3 \geq \min\{c_3|_{x=-1} = 16(2c+1-d), \quad c_3|_{x=1} = 16(-2c+1+d)\}$$

But,

$$\begin{aligned} 2c+1-d &= \frac{3c^2 + 2c(c+\alpha^2) + 2(c+\alpha^2) - 3}{2(c+\alpha^2)} \\ &> \frac{3c^2+2c-1}{2(c+\alpha^2)} > \frac{0.1}{2(c+\alpha^2)} \end{aligned}$$

since $0 < \theta < \theta_0$ and, hence, $c + \alpha^2 > 1$. On the other hand,

$$-2c+1+d = (1-c) + (d-c) = (1-c) + \frac{3}{2}\frac{1-c^2}{2(c+\alpha^2)} > 0$$

Hence, $c_3 > 0$.

The coefficient c_2 is quadratic in x. Let $v = v(\theta)$ denote the vertex of the quadratic c_2. Then, we have

$$\begin{aligned} c_2 \geq c_2|_{x=v} &= -\frac{2(-4c^4+9c^2+2d^2c^2-6dc+d^2-2)}{1+4c^2} \\ &= \frac{(1-c)^2(1+c)^2(14c^2+40\alpha^2c-9+8\alpha^4)}{2(c+\alpha^2)^2(1+4c^2)} \end{aligned}$$

However, we can the rewrite the term $14c^2 + 40\alpha^2 c - 9 + 8\alpha^4$

$$\begin{aligned} 14c^2 + 40\alpha^2c - 9 + 8\alpha^4 = \\ 8(c+\alpha^2)^2 - 8 + 6c(c+\alpha^2) + 18\alpha^2c - 1 > \\ 6c - 1 > 1.1 \end{aligned}$$

Clearly, c_1 and c_0 are non-negative. Hence, $p_4 \geq 0$.

Case 1.b $0 < \phi < \pi, \quad \theta_0 < \theta < \pi/2, \quad 0 < r < 1$

We need only show that $2 > h(r, \phi)$ for $2/5 < r < 1$, because Lemma 3.4 implies that for $0 < r < 2/5$, $h(r, \phi)$ has no critical points.

Let

$$4 - |h(r,\phi)|^2 = \frac{p_1(\theta, r, x)}{q_1(\theta, r, x)}$$

$x = \cos\phi$. It is easily seen that the numerator p_1 is a reflexive 8^{th}-degree polynomial in r. Making a change of variable $r = e^{-s}$, we can write

$$e^{4s}p_1(\theta, r, x) = p_2(\theta, \cosh s, x)$$

where p_2 is a 4^{th}-degree polynomial in $\cosh s$. We substitute $\cosh s = 1 + 2\sinh^2(s/2)$ into p_2 to obtain

$$p_3(\theta, \sinh(s/2), x) = p_2(\theta, 1 + 2\sinh^2(s/2), x)$$

which is an even 8^{th}-degree polynomial in $\sinh(s/2)$. Finally, we make a change of variable $\sqrt{t} = \sinh(s/2)$ to obtain

$$p_4(\theta, t, x) = p_3(\theta, \sinh(s/2), x)$$

which is a 4^{th}-degree polynomial in t.

We have reduced our problem to showing that

$$p_4(t) = p_4(\theta, t, x) = c_4(\theta, x)t^4 + c_3(\theta, x)t^3 + c_2(\theta, x)t^2 + c_1(\theta, x)t + c_0(\theta, x) > 0$$

for $0 < t < 225/1000$ under the assumption that $\theta_0 < \theta < \pi/2$, where

$$\begin{aligned} c_4 &= -16(c+\alpha^2+1)(c+\alpha^2-1) \\ c_3 &= (\alpha^2c^2 + 3\alpha^2 - c^3 + 2\alpha^4c - c)x - 2\alpha^4 + 4 - 4\alpha^2c - 2c^2 \\ c_2 &= (12c^2 + 8 - 4\alpha^4 - 8\alpha^2c)x^2 + (4\alpha^2c^2 - 36c + 8\alpha^4c + 12\alpha^2 - 4c^3)x \\ &- 7c^4 + 14c^2 - 12\alpha^2c + 4\alpha^2c^3 \\ c_1 &= (1-cx)(x-c)^2 \\ c_0 &= (x-c)^4 \end{aligned}$$

Since $\theta_0 < \theta < \pi/2$, we have $0 < c + \alpha^2 < 1$, which implies that $c_4 > 0$.

As in the previous case, c_3 is linear in x. Hence we have

$$c_3 \geq \min\{c_3|_{x=-1} = (1-c)q_m, \quad c_3|_{x=1} = (1+c)q_p\}$$

where

$$\begin{aligned} q_p &= c^2 - \alpha^2 c + 3c - 2\alpha^4 + 3\alpha^2 + 4 \\ q_m &= c^2 - \alpha^2 c - 3c - 2\alpha^4 - 3\alpha^2 + 4 \end{aligned}$$

We can rewrite $q_p = c^2 + 3(c+\alpha^2) + 2 + (1 - \alpha^2(c+\alpha^2)) + (1-\alpha^4) > 0$ since $0 < c + \alpha^2 < 1$ and $0 < \alpha < 1$.

On the other hand, the estimate for q_m is more delicate. Making a change of variable $c = 2y^2 - 1$ (where $y = \cos\theta$), we have

$$q_m = 4y^4 - 10y^2 + 8 - 2\alpha^2 y^2 - 2\alpha^2 - 2\alpha^4.$$

Note that all of the coefficients of α in q_m are negative. Since we are looking for a lower bound for q_m, we replace $\alpha = \alpha(y)$ by an upper bound obtained from Lemma 3.2. Specifically, we bound α by the 8^{th}-order partial sum

$$\alpha \leq p_8 = 1 - \frac{1}{4}y^2 - \frac{1}{16}y^4 - \frac{7}{192}y^6 - \frac{19}{768}y^8.$$

We have then,

$$q_m \geq q_m^* = q_m|_{\alpha=p_8},$$

The polynomial q_m^* is a 32^{nd} degree polynomial in y with rational coefficients. A Sturm sequence argument shows that q_m^* has no roots on (0,1]. Hence, $q_m > 0$.

Clearly, c_1 and c_0 are non-negative.

However, c_2 is not. Consequently, it is not immediately obvious that $p_4(t) > 0$ for $0 < t < 225/1000$. Let

$$q = q(t) = c_3(\theta, x)t^2 + c_2(\theta, x)t + c_1(\theta, x).$$

We will show that $q(t) > 0$ for $0 < t < 1/4$, which will imply that $p_4(t) > 0$ for $0 < t < 225/1000$.

We note that it can be shown that $c_2 = c_2(\theta, x)$ is non-negative for $\theta_0 < \theta < \pi/2$ and $-4/5 < x < 1$. Hence, we will show that $q(t) > 0$ for $\theta_0 < \theta < \pi/2$, $-1 < x < -4/5$, and $0 < t < 1/4$.

Expanding q in powers of α we have

$$\begin{aligned} (3.7) \quad q &= [-4(x-c)^2 t - 16(1-cx)t^2]\alpha^4 \\ &+ [(4c^3 - 12c + 4c^2 x + 12x - 8x^2 c)t + (-24c + 24x + 8c^2 x)t^2]\alpha^2 \\ &- 8ct^2(c+\alpha^2) + \tilde{q}(c, x, t) \end{aligned}$$

where $\tilde{q} = \tilde{q}(c, x, t)$ is quadratic in t and independent of α.

Clearly the coefficient of α^4 negative. It is relatively straightforward to verify that each of the components (the coefficients of t and t^2) of the coefficient of α^2 are negative. However, the sign of term $-8ct^2(c+\alpha^2)$ depends on the sign of c.

Case 1-b-1 $\theta_0 < \theta < \pi/4$.

In this case $c > 0$; we will replace $(c+\alpha^2)$ in (3.7) by an upper bound 1. Hence,

$$\begin{aligned} q > q_0 &= [-4(x-c)^2 t - 16(1-cx)t^2]\alpha^4 \\ &+ [(4c^3 - 12c + 4c^2 x + 12x - 8x^2 c)t + (-24c + 24x + 8c^2 x)t^2]\alpha^2 \\ &- 8ct^2 + \tilde{q}(c,x,t) \end{aligned}$$

Since the coefficients of α in q_0 are negative, we will replace α by an upper bound from Lemma 3.2, namely $\alpha \leq p_2 = 1 - \frac{y^2}{4}$. Hence, we have, making a change of variable $c = 2y^2 - 1$,

$$q_0 \geq q_0^*(t) = q_0|_{\alpha=p_2}(t) = d_2(y,x)t^2 + d_1(y,x)t + d_0(y,x)$$

where

$$\begin{aligned} d_2(y,x) &= (\frac{1}{8}y^{10} - \frac{1}{16}y^8 - 69y^6 + 108y^4 - 64y^2 + 32)x \\ &- \frac{1}{16}y^8 - 2y^6 - \frac{25}{2}y^4 - 28y^2 + 40 \\ d_1(y,x) &= (-\frac{1}{64}y^8 - \frac{3}{4}y^6 + 55y^4 - 64y^2 + 24)x^2 \\ &+ (\frac{1}{16}y^{10} - \frac{1}{32}y^8 - \frac{69}{2}y^6 + 54y^4 - 96y^2 + 48)x \\ &- \frac{1}{16}y^{12} + \frac{49}{16}y^{10} - \frac{2689}{64}y^8 + \frac{441}{4}y^6 - 49y^4 - 32y^2 + 24 \\ d_0(y,x) &= (-16y^2 + 8)x^3 + (64y^4 - 64y^2 + 24)x^2 \\ &+ (-64y^6 + 96y^4 - 80y^2 + 24)x + 32y^4 - 32y^2 + 8. \end{aligned}$$

The coefficents $d_j(y,x),\ \ j = 0,\ 1,\ 2$ are polynomials with rational coefficients in y, x subject to the parameter restrictions that $\cos\pi/4 < y < \cos\theta_0$ and $-1 < x < -4/5$. It can be shown using Lemma 3.3 that $d_0(y,x) > 0$ and $d_1(y,x) > 0$ for these restricted parameters. Hence, $q_0^*(t) > 0$ if $q_0^*(1/4) > 0$. However, $q_0^*(1/4)$ is a polynomial with rational coefficients in y, x subject to the same parameter restrictions that $\cos\pi/4 < y < \cos\theta_0$ and $-1 < x < -4/5$. Using Lemma 3.3 it can be shown that $q_0^*(1/4) > 0$.

Case 1-b-2 $\pi/4 < \theta < \pi/2$

In this case $c < 0$; we will replace $(c+\alpha^2)$ in (3.7) by a lower bound $(1+c)/2$. Hence,

$$\begin{aligned} q > q_0 &= [-4(x-c)^2 t - 16(1-cx)t^2]\alpha^4 \\ &+ [(4c^3 - 12c + 4c^2 x + 12x - 8x^2 c)t + (-24c + 24x + 8c^2 x)t^2]\alpha^2 \\ &- 8ct^2(1+c)/2 + \tilde{q}(c,x,t) \end{aligned}$$

Since the coefficients of α in q_0 are negative, we will replace α by an upper bound of 1. Hence, we have, making a change of variable $c = 2y^2 - 1$,

$$q_0 \geq q_0^*(t) = q_0|_{\alpha=1}(t) = e_2(y,x)t^2 + e_1(y,x)t + e_0(y,x)$$

Furthermore, for convenience in scaling we will impose a change of variable $x = -1 + 2w/10$, where $0 < w < 1$. We have then

$$q_0^*(t) = q_0^*(y,w,t) = e_2(y,w)t^2 + e_1(y,w)t + e_0(y,w)$$

where $0 < y < \cos\theta_0$, $0 < w < 1$ and $0 < t < 1/44$ and

$$\begin{aligned} e_2(y,w) &= (-\frac{64}{5}y^6 + \frac{128}{5}y^4 - \frac{64}{5}y^2 + \frac{32}{5})w \\ &+ 64y^6 - 176y^4 + 56y^2 \\ e_1(y,w) &= (\frac{48}{25}y^4 - \frac{64}{25}y^2 + \frac{24}{25})w^2 \\ &+ (-\frac{32}{5}y^6 - \frac{32}{5}y^4 + \frac{32}{5}y^2)w \\ &- 16y^8 + 96y^6 - 48y^4 \\ e_0(y,w) &= (-\frac{16}{125}y^2 + \frac{8}{125})w^3 + (\frac{64}{25}y^4 - \frac{16}{25}y^2)w^2 \\ &+ (-\frac{64}{5}y^6 - \frac{32}{5}y^4)w + 64y^6 \end{aligned}$$

We note that using Lemma 3.3 it can be verified that $q_0^*(0) = e_0(y,w)$ is non-negative and that $q_0^*(1/4)$ is also non-negative.

Let R be the parameter region for $y,\ w$, i.e, $R = \{(y,w) : 0 < y < \sqrt{2}/2,\ 0 < w < 1\}$ We will partition R into subregions bounded by curves $l_j,\ j = 1,\ 2,\ 3$. See Figure 3.

The first curve l_1 is defined as the solution set $\{(y,w) : e_2(y,w) = 0\}$ and, since $e_2(y,w)$ is linear in w, is given by

$$l_1 = \{(y,w) \in R : w = \frac{5y^2(8y^4 - 22y^2 + 7)}{4(2y^6 - 4y^4 + 2y^2 - 1)}\}$$

Let A be the subset of R to the "right" of l_1, i.e., the subset of R where $e_2(y,w) < 0$. On A, because $q_0^*(t)$ is concave down, it suffices to check that $q_0^*(0) > 0$ and $q_0^*(1/4) > 0$ to verify that $q_0^*(t) > 0$.

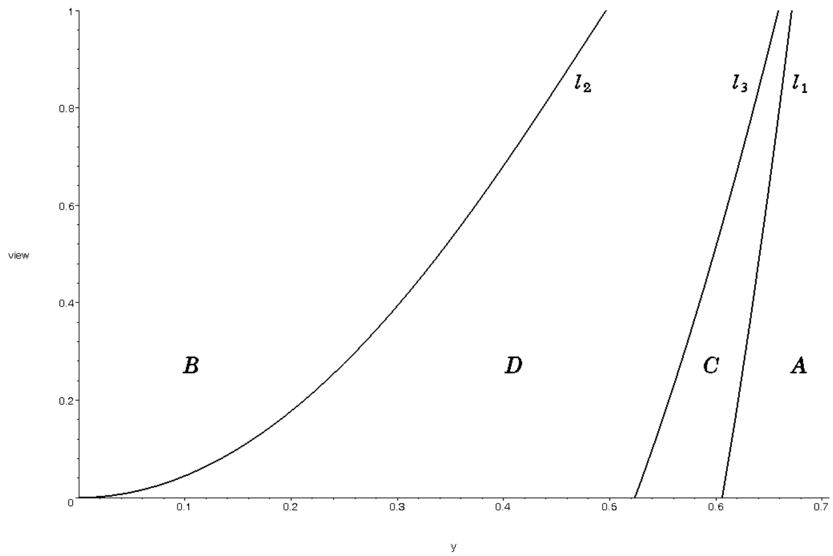

FIGURE 3. The parameter space R.

The second curve l_2 is defined as the solution set $\{(y, w) : e_1(y, w) = 0\}$ and, since $e_1(y, w)$ is quadratic in w, is given by

$$l_2 = \{(y, w) \in R : w = y^2 \frac{20y^4 + 20y^2 - 20 + 10\sqrt{16y^8 - 80y^6 + 134y^4 - 92y^2 + 22}}{2(6y^4 - 8y^2 + 3)}\}$$

Let B be the subset of R to the "left" of l_2, i.e., the subset of R where $e_1(y, w) > 0$. On B, because $q_0^*(t)$ is concave up and because the slope to $q_0^*(t)$ at 0 is positive, it suffices to check that $q_0^*(0) > 0$ to verify that $q_0^*(t) > 0$.

The third curve l_3 is defined as the solution set $\{(y, w) : e_2(y, w)/2+e_1(y, w) = 0\}$, i.e., the set where the vertex of $q_0^*(t)$ is located at $t = 1/4$. Since $e_2(y, w)/2 + e_1(y, w)$ is quadratic in w, l_3 is given by

$$l_3 = \{(y, w) \in R : w = y^2 \frac{80y^6 - 40y^4 - 20 + 10\sqrt{k(y)}}{2(12y^4 - 16y^2 + 6)}\}$$

where $k(y) = 112y^{12} - 512y^{10} + 960y^8 - 852y^6 + 332y^4 - 42y^2 + 4$. Let C be the subset of R bounded between l_3 and l_1, i.e., the set where $q_0^*(t)$ is concave up and the vertex of $q_0^*(t)$ is located to the right of $t = 1/4$. On C it suffices to check that $q_0^*(1/4) > 0$ to verify that $q_0^*(t) > 0$.

Finally, let D be the subset of R bounded between l_2 and l_3. On D, the quadratic $q_0^*(t)$ is concave up and the vertex lies between $t = 0$ and $t = 1/4$. To verify that $q_0^*(t) > 0$, we need to verify that $q_0^*(t)|_{t=\text{vertex}} > 0$ on D or alternatively that the discriminant of $q_0^*(t)$ is negative on D.

Ideally, to solve this latter problem one would represent $y \in D$ in terms of a convex average of values on l_2 and l_3. However, the curves l_2, l_3 which bound D are inconvenient to work with. Instead, we will bound l_2 and l_3 by (approximating) polynomial curves, m_2 and m_3, which lie outside of D and show

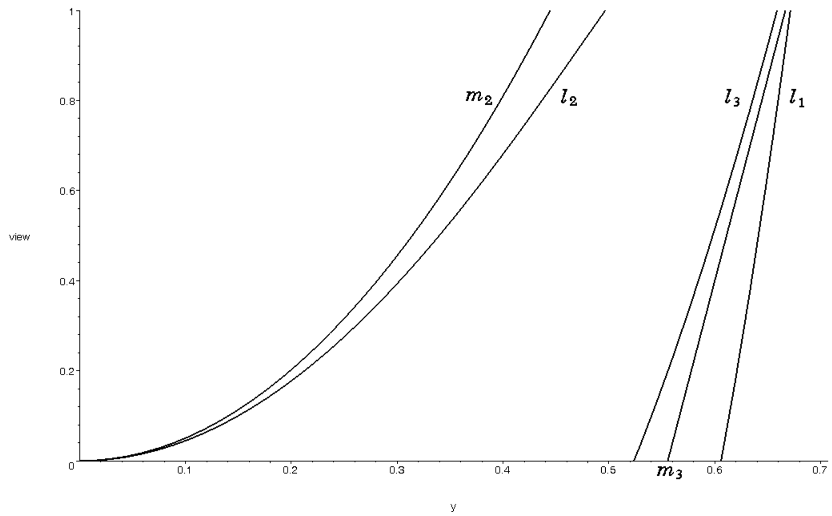

FIGURE 4. The bounding curves m_2 and m_3.

that the discriminant of $q_0^*(t)$ is negative on the region $D*$ which is bounded by our approximating curves m_2 and m_3 and which contains D.

Let $m_2 = \{(y, w) \in R : w = \frac{81}{16}y^2\}$ and $m_3 = \{(y, w) \in R : w = 9y - 5\}$. Let D^* be the subset of R bounded between m_2 and m_3. See Figure 4. To show that m_2 lies to the left of l_2, one isolates the square root in the inequality $m_2 - l_2 > 0$, squares the terms in the resulting inequality and then collects all of the terms to obtain a new inequality of the form

$$p(y) = 3876y^8 - 14816y^6 + 58020y^4 - 62448y^2 + 21609 > 0$$

A Sturm sequence argument then verifies $p(y) > 0$ on the interval $0 < y < \sqrt{2}/2$. A similar argument show that m_3 lies to the right of l_3. Consequently, $D^* \supset D$.

Let $y_2(w) = \frac{4}{9}\sqrt{w}$ be the inverse function for m_2 and $y_3(w) = \frac{1}{9}w + \frac{5}{9}$ be the inverse function for m_3 and let $w = v^2$. Define now

$$\begin{aligned} f(z, v, t) &= q_0^*(y_2(v^2) + z(y_3(v^2) - y_2(v^2)), v^2, t) \\ &= f_2(z, v)t^2 + f_1(z, v)t + f_0(z, v) \end{aligned}$$

We have then that f is a polynomial in z, v, t with rational coefficients. As a quadratic in t, it sufficies to show that the discriminant $f(t)$ is negative for $0 < z < 1$, $0 < v < 1$. Let

$$g(z, v) = 4f_2(z, v)f_0(z, v) - (f_1(z, v))^2$$

There is a complication that at $(z, v) = (0, 0)$ the function g has a higher order zero. We will partition the parameter square [0,1]x[0,1] into triangles

$$\begin{aligned} T_l &= \{(z, v) : z = mv,\ 0 < v < 1,\ 0 < m < 1\} \\ T_u &= \{(z, v) : v = mz,\ 0 < v < 1,\ 0 < m < 1\} \end{aligned}$$

and then set

$$\begin{aligned} g_l(m,v) &= g(mv,v),\ 0<v<1,\ 0<m<1 \\ g_u(z,m) &= g(z,mz),\ 0<z<1,\ 0<m<1 \end{aligned}$$

The polynomial g_l can be factored as

$$g_l(m,v) = v^8 h_l(m,v)$$

where h_l is a polynomial in $m,\ v$ with rational coefficients. The polynomial h_l is of degree 16 in m and degree 40 in v and satisfies $h_l(0,0)>0$.

We apply now Lemma 3.3 to h_l. Explicitly, for the region $R=\{(m,v): 0<m<1,\ 0<v<1\}$, let $L=L(12000,12000)$. Writing

$$h_l(v,m)=\sum_{i=0}^{16} u_i(v)m^i$$

we have

$$\frac{\partial h_l}{\partial v}(v,m)=\sum_{i=0}^{16} {u_i}'(v)m^i,\quad \frac{\partial h_l}{\partial m}(v,m)=\sum_{i=1}^{16} iu_i(v)m^{i-1}.$$

Hence,

$$\left|\frac{\partial h_l}{\partial v}(m,v)\right| \le M_1=698,\ \left|\frac{\partial h_l}{\partial m}(m,v)\right| \le M_2=872$$

where

$$\begin{aligned} M_1 &\ge \sum_{i=0}^{16}\mu_i,\quad \mu_i=\max_{0\le v\le 1}|{u_i}'(v)| \\ M_2 &\ge \sum_{i=1}^{16}\nu_i,\quad \nu_i=\max_{0\le v\le 1}|iu_i(v)|. \end{aligned}$$

Set $M=\max\{M_1,M_2\}=872$. A lengthy finite-arithmetic calculation of the values of polynomial h_l over the lattice L yields

$$m=\min_{(m,v)\in L} h_l(m,v)=\frac{2158649303}{26904200625}\approx 0.080.$$

By Lemma 3.3, we have $h_l(m,v)\ge 0$ on R.

A similar argument show that $g_u(z,m)$ is non-negative on $0<z<1,\ 0<m<1$.

Case 2. $\phi=\pi,\quad 0<\theta<\pi/2,\quad 0<r<1$

Let

$$g(r)=(1-r)^2 S_{f_\alpha}(-r)=(1-r)^2 2(c+\alpha^2)\frac{1+2dr+r^2}{(1+2cr+r^2)^2}$$

where $0<r<1$. For fixed α we have

$$g'(r)=4(1-r^2)(c+\alpha^2)\frac{(d-2c-1)r^2-2(dc+2d-c)r+d-2c-1}{(1+2cr+r^2)^3}$$

The sign of g' is determined by the sign of

$$(d-2c-1)(r^2 - 2\frac{dc+2d-c}{d-2c-1}r + 1).$$

Since d is increasing as a function of θ and $2c+1$ is decreasing as a function of θ, then the factor $d-2c-1$ has a unique root θ_1 such that $d-2c-1 < 0$ for $0 < \theta < \theta_1$ (and $d-2c-1 > 0$ for $\theta_1 < \theta < \pi/2$). Numerically, $\theta_1 \approx 0.598$. Using a lower estimate for α from Lemma 3.2, it can be shown that $\cos\theta_1 < 83/100$. Note, $\theta_1 > \theta_0$.

It is easily verified that $dc + 2d - c = c(d-1) + 2d > 0$ for $0 < \theta < \theta_1$. Hence, the coefficient $-2\frac{dc+2d-c}{d-2c-1} > 0$ for $0 < \theta < \theta_1$ which implies that g' is negative for $0 < \theta < \theta_1$. Hence, g takes its maximum at 0 for $0 < \theta < \theta_1$, but the value $g(0)$ is $2(c+\alpha^2)$, which is covered in Case 3.

We will show that for $\theta_1 < \theta < \pi/2$ that $2 > g(r)$. Since we show in Case 3 that the maximal value for the Schwarz norm of f_α is more than 2, then we will have that no value of $g(r)$ for $\theta_1 < \theta < \pi/2$ can be extremal for our problem of maximizing the Schwarz norm of f_α.

Let

$$2 - g(r) = \frac{p_1(\theta, r)}{q_1(\theta, r)}.$$

It is easily seen that the numerator p_1 is a reflexive 4^{th}-degree polynomial in r. Making a change of variable $r = e^{-s}$, we can write

$$e^{2s} p_1(\theta, r) = p_2(\theta, \cosh s)$$

where p_2 is a 2^{nd}-degree polynomial in $\cosh s$. We substitute $\cosh s = 1 + 2\sinh^2(s/2)$ into p_2 to obtain

$$p_3(\theta, \sinh(s/2)) = p_2(\theta, 1 + 2\sinh^2(s/2))$$

which is an even 4^{th}-degree polynomial in $\sinh(s/2)$. Finally, we make a change of variable $\sqrt{t} = \sinh(s/2)$ to obtain

$$p_4(\theta, t) = p_3(\theta, \sinh(s/2))$$

which is a 2^{nd}-degree polynomial in t.

We have reduced our problem to showing that

$$p_4(t) = p_4(\theta, t) = c_2(\theta)t^2 + c_1(\theta)t + c_0(\theta) > 0$$

for $t > 0$ under the assumption that $\theta_1 < \theta < \pi/2$ where

$$\begin{aligned} c_2 &= 8[1 - (c + \alpha^2)] \\ c_1 &= 4(2 - \alpha^2 + c - dc - \alpha^2 d) \\ c_0 &= 2(1+c)^2. \end{aligned}$$

Let v denote the vertex of p_4. Then

$$p_4|_{t=v} = \frac{c+\alpha^2}{2[1-(c+\alpha^2)]} q_1$$

where

$$q_1 = \frac{(1+c)^2}{4(c+\alpha^2)} q_2$$

with

$$q_2 = -4\alpha^4 + (4c-12)\alpha^2 - c^2 - 18c + 15.$$

Making a change of variable $c = 2y^2 - 1$ (where $y = \cos\theta$), we have

$$q_2 = -4\alpha^4 + (8y^2-16)\alpha^2 - 4y^4 - 32y^2 + 32.$$

Note that all of the coefficients of α in q_2 are negative. Since we are looking for a lower bound for q_2, we replace $\alpha = \alpha(y)$ by an upper bound obtained from Lemma 3.2. Specifically, we bound α by the 8^{th}-order partial sum

$$\alpha \le p_8 = 1 - \frac{1}{4}y^2 - \frac{1}{16}y^4 - \frac{7}{192}y^6 - \frac{19}{768}y^8.$$

We have then,

$$q_2 \ge q_2^* = q_2|_{\alpha=p_8},$$

The polynomial q_2^* is a 32^{nd}-degree polynomial in y with rational coefficients. A Sturm sequence argument shows that q_2^* has no roots on (0,83/100]. Hence, $q_2 > 0$.

References

[1] G.D. Anderson, M.K. Vamanamurthy, and M.K. Vuorinen, *Conformal invariants, inequalities, and quasiconformal mappings*, J. Wiley and Sons, 1997.

[2] R. W. Barnard, L. Cole, K. Pearce, and G. Brock Williams, *The sharp bound for the deformation of a disc under a hyperbolically convex map*, Proceedings of the London Mathematical Society (accepted).

[3] A. Beardon, *Geometry of discrete groups*, Springer-Verlag, Berlin, 1983.

[4] Frederick P. Gardiner and Nikola Lakic, *Quasiconformal Teichmüller theory*, Mathematical Surveys and Monographs, vol. 76, American Mathematical Society, 2000.

[5] O. Lehto, *Univalent functions and Teichmüller spaces*, Springer-Verlag, Berlin - Heidelberg - New York, 1987.

[6] William Ma and David Minda, *Hyperbolically convex functions*, Ann. Polon. Math. **60** (1994), no. 1, 81–100. MR **95k:**30037

[7] ______, *Hyperbolically convex functions. II*, Ann. Polon. Math. **71** (1999), no. 3, 273–285. MR **2000j:**30020

[8] Diego Mejía and Christian Pommerenke, *On hyperbolically convex functions*, J. Geom. Anal. **10** (2000), no. 2, 365–378.

[9] ______, *On spherically convex univalent functions*, Michigan Math. J. **47** (2000), no. 1, 163–172. MR **2001a:**30013

[10] ______, *Sobre la derivada Schawarziana de aplicaciones conformes hiperbólicamente*, Revista Colombiana de Matemáticas **35** (2001), no. 2, 51–60.

[11] ______, *Hyperbolically convex functions, dimension and capacity*, Complex Variables Theory Appl. **47** (2002), no. 9, 803–814. MR **2003f:**30017

[12] ______, *On the derivative of hyperbolically convex functions*, Ann. Acad. Sci. Fenn. Math. **27** (2002), no. 1, 47–56.

[13] Diego Mejía, Christian Pommerenke, and Alexander Vasil′ev, *Distortion theorems for hyperbolically convex functions*, Complex Variables Theory Appl. **44** (2001), no. 2, 117–130. MR **2003e:**30025

[14] Zeev Nehari, *The Schwarzian derivative and schlicht functions*, Bull. Amer. Math. Soc. **55** (1949), 545–551. MR 10,696e

[15] ———, *A property of convex conformal maps*, J. Analyse Math. **30** (1976), 390–393. MR 55 #12901

[16] A. Yu. Solynin, *Some extremal problems on the hyperbolic polygons*, Complex Variables Theory Appl. **36** (1998), no. 3, 207–231. MR **99j:**30030

[17] ———, *Moduli and extremal metric problems*, Algebra i Analiz **11** (1999), no. 1, 3–86, translation in St. Petersburg Math. J. **11** (2000), no. 1, 1–65. MR **2001b:**30058

Roger W. Barnard and Kent Pearce E-MAIL: `roger.w.barnard@ttu.edu`
E-MAIL: `kent.pearce@ttu.edu`
ADDRESS: *Department of Mathematics and Statistics, Texas Tech University, Lubbock, TX 79409–1042*