

Answer the problems on separate paper. You do not need to rewrite the problem statements on your answer sheets. Work carefully. Do your own work. **Show all relevant supporting steps!**

1. (12 pts) Give the definition for each of the following:
  - a. A set  $S$  in  $\mathbb{C}$  is starlike with respect to a point  $a \in S$  if . . .
  - b. A region  $G$  is simply connected if . . .
  - c. A function  $f$  has a branch-of-log  $f$  on region  $G$  if . . .
2. (24 pts) State each of the following theorems, giving attention to the correct hypotheses for each:
  - a. Independence of Path Theorem
  - b. Casorati-Weierstrass Theorem
  - c. Residue Theorem
3. (28 pts) Identify each of the finite isolated singularities for each of the following functions. Determine whether the singularities are removable singularities, poles or essential singularities. If the singularity is removable, determine what value the function should be assigned at the singularity to analytically extend the function at the singularity. If the singularity is a pole, determine the order of the pole. If the isolated singularity occurs at  $z = 0$ , then determine residue of the singularity.
  - a. (10)  $f(z) = \frac{\sin \pi z}{z^2(z+1)}$
  - b. (6)  $f(z) = (z^3 - z)e^{1/z}$
  - c. (6)  $f(z) = \frac{\log z}{z(z^2 + 1)}$
  - d. (6)  $f(z) = z^2 \left( \frac{1}{z} - \sin \frac{1}{z} \right)$
4. (20 pts) Prove the following corollary to Cauchy's Theorem (Version #4):

Corollary. Let  $G$  be a region in  $\mathbb{C}$  which is simply connected and let  $f \in \mathcal{A}(G)$ . If  $f(z) \neq 0$  on  $G$ , then  $f$  has a branch-of-log  $f$  on  $G$ .

5. (21 pts) Classify each of the following regions as to whether they are convex or not convex, starlike (with respect to some point  $a \in G$ ) or not starlike (with respect to any point  $a \in G$ ), simply connected or not simply connected.

Notation:  $[a,b]$  denotes the (closed) straight line segment between  $a$  and  $b$ .

$\mathcal{Q}_1$  denotes the first quadrant.

Use the table on the next page to report your answers

N.B. Mark each cell in the table as Y or N, i.e., do not leave any cell in the table blank.

- a.  $G_a = B(0,1) \setminus [-\frac{1}{2}, 1)$       b.  $G_b = B(0,1) \setminus \overline{B(1, \sqrt{2})}$
- c.  $G_c = B(0,1) \setminus \{[-\frac{1}{2}, \frac{1}{2}] \cup [-\frac{i}{2}, \frac{i}{2}]\}$       d.  $G_d = B(0,1) \setminus \{0\}$
- e.  $G_e = B(0,1) \cap B(1, \sqrt{2})$       f.  $G_f = B(0,1) \setminus \overline{B(\frac{1+i}{4}, \frac{1}{4})}$
- g.  $G_g = B(0,1) \setminus \overline{B(\frac{1+i}{2}, \frac{1}{2})}$       h.  $G_h = B(0,1) \cap B(2, 2)$
- i.  $G_i = H_i \setminus \overline{\{B(-1, \frac{1}{2}) \cup B(1, \frac{1}{2})\}}$  where  $H_i = \{z : |\operatorname{Re} z| < 1\}$
- j.  $G_j = H_j \setminus \overline{\{B(-1, 1) \cup B(1, 1)\}}$  where  $H_j = \{z : |\operatorname{Re} z| < 1 \text{ and } \operatorname{Im} z > 0\}$
- k.  $G_k = B(0,1) \setminus H_k$  where  $H_k = \{z : \operatorname{Re} z \leq -\frac{1}{2}\}$
- l.  $G_l = \mathcal{Q}_1 \setminus \{[1, 1+i] \cup [2, 2+i]\}$
- m.  $G_m = S_m \setminus \overline{\{B(1+i, 1) \cup B(1-i, 1) \cup B(-1-i, 1) \cup B(-1+i, 1)\}}$  where  
 $S_m = \{z : |\operatorname{Re} z| < 1 \text{ and } |\operatorname{Im} z| < 1\}$
- n.  $G_n = \mathcal{Q}_1 \setminus \{[3, 3+i] \cup [2i, 2i+1]\}$

Name \_\_\_\_\_

(Return this sheet with your other problems)

	Convex	Starlike	Simply Connected		Convex	Starlike	Simply Connected
$G_a$				$G_h$			
$G_b$				$G_i$			
$G_c$				$G_j$			
$G_d$				$G_k$			
$G_e$				$G_l$			
$G_f$				$G_m$			
$G_g$				$G_n$			