

Notation:

$$D = \{ z \in \mathbb{C} \mid |z| < 1 \}$$

$$G = \text{region in } \mathbb{C}$$

$$C(G, \mathbb{C}) = \{ f \mid f \text{ is continuous on } G, f: G \rightarrow \mathbb{C} \}$$

$$A(G) = \{ f \mid f \text{ is analytic on } G \} \quad H(G) = \{ u \mid u \text{ is harmonic on } G \}$$

Each problem or part is worth 20 points.

1. Compute $\int_0^{\infty} \frac{1}{1+x^5} dx$.

2. Prove that $\int_1^{\infty} \frac{t^z - 1}{e^t - 1} dt$ is analytic on $\text{Re } z > 1$.

3. Let $f \in A(D)$. Suppose that f has only a finite number of zeros on D . Prove there exists a $g \in A(D)$ and a polynomial p such that $f = e^g p$ on D .
4. There are many parallels between analytic and harmonic functions: both satisfy Laplace's equation; both are infinitely differentiable; both satisfy a mean value property; both are open mappings; both satisfy a maximum principle; etc.

Are the following properties also parallel?

(a) The sum of two such functions [in $A(G)$ or $H(G)$] is again such a function [in $A(G)$ or $H(G)$]?

(b) The product of two such functions [in $A(G)$ or $H(G)$] is again such a function [in $A(G)$ or $H(G)$]?

(c) For non-vanishing such functions [in $A(G)$ or $H(G)$] the reciprocal is again such a function [in $A(G)$ or $H(G)$]?

5. In this problem, if you use an infinite sum or product, you must prove that it converges, i.e., it meets an appropriate set of conditions to ensure convergence.

Find an explicit function $f \in A(\mathbb{C})$ with simple zeros at $\sqrt{n}$, $n = 1, 2, 3, \dots$, and no other zeros.

6. For $g \in A(\mathbb{C})$, let $Z(g)$ denote the zero set of g . Suppose that $\{f_n\} \subset A(\mathbb{C})$ and that $Z(f_n) \subset \mathbb{R}$ for each n . Suppose there exists $f \in C(\mathbb{C}, \mathbb{C})$ such that $f_n \rightarrow f$ locally uniformly on compact subsets of $\mathbb{C}$. Show that $f \in A(\mathbb{C})$ and that if $f \neq 0$, then $Z(f) \subset \mathbb{R}$.
7. Let $p_n(z) = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \dots + \frac{z^n}{n!}$, $n = 1, 2, 3, \dots$. Prove for each $r > 0$, there exists an integer $N = N(r)$ such that whenever $n > N$, then $Z(p_n) \cap B(0, r) = \emptyset$.
8. Find the maximal region G in $\mathbb{C}$ such that $\sum_{n=1}^{\infty} \frac{1}{n^{i(z^2+1)}}$ converges in $A(G)$.