Review Exam III

Complex Analysis

Underlined Propositions or Theorems: Proofs May Be Asked for on Exam

Chapter 3.3

Bi-Linear, Linear Fractional, Moebius Transformations

Definition $S(z) = \dfrac{az+b}{cz+d}$, $ad - bc \neq 0$

Differentiability $S'(z) = \dfrac{ad-bc}{(cz+d)^2}$

Properties $S : \mathbb{C}_\infty \to \mathbb{C}_\infty$, S is 1-1, S is onto, pole of S is $-d/c$, zero of S is $-b/a$

$T(z) = S^{-1}(z) = \dfrac{-dz+b}{cz-a} = \dfrac{dz-b}{-ca+a}$, S conformal on $\mathbb{C} \setminus \{-d/c\}$, Fixed Points, Uniqueness

Special Cases: Translations, Dilations, Rotations, Inversion
Every Bi-Linear Transformation can be written as a composition of Translations, Dilations, Rotations, and the Inversion

Mapping Properties of Special Cases: “Circles” to “Circles”
Geometry of Images

Cross-Ratio

Definition: $S(z) = (z, z_2, z_3, z_4)$ unique Bi-Linear Transformation such that $S(z_2) = 1$, $S(z_3) = 0$, $S(z_4) = \infty$

Proposition: Cross-Ratio in Invariant under Bi-Linear Transformations

Properties Orientation Principle, Symmetry

Joukowski Transformation

Conformal Mappings

Compositions of Standard Functions, Bi-Linear Transformations and Joukowski

<u>Examples</u> of conformal mappings
<u>Problems</u> about constructing conformal mappings

Chapter 4.1

Riemann-Stieltjes Integrals

Definition of function of bounded variation and total variation

Proposition (1.3) If $g : [a,b] \to \mathbb{C}$ is piecewise smooth, then g is of bounded variation and $V(g) = \int_a^b |g'(t)|\, dt$.

Definition of Riemann-Stieltjes Integral

Theorem (1.4) If $f : [a,b] \to \mathbb{C}$ is continuous and if $g : [a,b] \to \mathbb{C}$ is of bounded variation, then the Riemann-Stieltjes integral $\int_a^b f\ dg = \int_a^b f(t)dg(t)$ exists.

(Proof uses Cantor's Theorem II.3.7).

<u>Proposition 1.7</u> Let $f, g : [a,b] \to \mathbb{C}$ be continuous, let $g, s : [a,b] \to \mathbb{C}$ be of bounded variation and let $a, b \in \mathbb{C}$. Then,

a) $\int_a^b (a f + b g)dg = a \int_a^b f dg + b \int_a^b g dg$

b) $\int_a^b fd(ag + b s) = a \int_a^b f dg + b \int_a^b g d s$

Proposition Let $f : [a,b] \to \mathbb{C}$ be continuous and let $g : [a,b] \to \mathbb{C}$ be of bounded variation. If $a < t_0 < t_1 < \cdots < t_n = b$, then $\int_a^b fdg = \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} fdg$

<u>Theorem (1.9)</u> If $g : [a,b] \to \mathbb{C}$ is piecewise smooth and $f : [a,b] \to \mathbb{C}$, then $\int_a^b f\ dg = \int_a^b f(t)g'(t)dt$.

Definition for a path $g : [a,b] \to \mathbb{C}$ of trace of g, $\{g\}$.

Definition of rectifiable path $g : [a,b] \to \mathbb{C}$ and length of $\{g\} = \int_a^b dg$. For g piece-wise smooth, length of $\{g\} = \int_a^b |g'(t)|\, dt$.

Definition of line integral: Let $g : [a,b] \to \mathbb{C}$ be rectifiable path and let $f : \{g\} \to \mathbb{C}$ be continuous, define line integral $\int_g f = \int_a^b (f \circ g) dg = \int_a^b f(g(t)) dg(t) = \int_g f(z) dz$

Note: if g piece-wise smooth, then $\int_g f = \int_a^b (f \circ g) dg = \int_a^b f(g(t)) dg(t) = \int_a^b f(g(t)) g'(t) dt = \int_g f(z) dz$

<u>Problems</u> about computing line integrals using the definition

Definition of a change of parameter j

Proposition If j is a change of parameter, i.e., if $j : [c,d] \to [a,b]$, j is continuous, strictly increasing and j is onto, then for $g : [a,b] \to \mathbb{C}$ a rectifiable path and $f : \{g\} \to \mathbb{C}$ continuous, then $\int_g f = \int_{g \circ j} f$

Definition: (1) a curve as an equivalance class of rectifiable paths;
(2) the trace of a curve is the trace of a representative;
(3) a curve is smooth if some representative is smooth;
(4) a curve is closed if the initial and terminal points on the trace are the same.

Definition for $g : [a,b] \to \mathbb{C}$ a rectifiable path of $-g$ and of $|g(t)|$ and definition

$$\int_g f(z) |dz| = \int_a^b f(g(t)) d|g|(t)$$

<u>Proposition (1.17)</u> Let $g : [a,b] \to \mathbb{C}$ be a rectifiable path and let $f : \{g\} \to \mathbb{C}$ be continuous. Then,

a) $\int_{-g} f = -\int_g f$

b) $|\int_g f| \le \int_g |f| |dz| \le \max_{z \in \{g\}} |f(z)| V(g)$

Theorem (Fundamental of Theorem of Calculus for Line Integrals) Let G be a region and let g be a rectifiable path in G with initial and terminal points a and b, resp. If $f : G \to \mathbb{C}$ is continuous and if f has a primitive on G, say F, then $\int_g f = F(z)\big|_a^b$.

Corollary Let G be a region and let g be a closed rectifiable path in G. If $f : G \to \mathbb{C}$ is continuous and if f has a primitive on G, say F, then $\int_g f = 0$.

<u>Problems</u> about computing line integrals using the Fund. Thm. of Calc. for Line Integrals

Chapter 4.2

Proposition 2.1 (Leibnitz's Rule)

Integrals

a) $$\int_{|w|=1} (w-z)^n dw = 0, \quad n = 0,1,2,3,\cdots$$

b) $$\int_{|w|=1} \frac{dw}{(w-z)^n} = 0, \quad \begin{cases} n = 2,3,4,5,\cdots \\ \quad |z| \neq 1 \end{cases}$$

c) $$\int_{|w|=1} \frac{dw}{w-z} = \begin{cases} 0, & |z|>1 \\ 2p\,i, & |z|<1 \end{cases}$$

Cauchy Integral Formula #0 Let $f : G \to \mathbb{C}$ be analytic and suppose that $\overline{B(a,r)} \subset G$. For $z \in B(a,r)$,

$$f(z) = \frac{1}{2p\,i} \int_{|w-a|=r} \frac{f(w)}{w-z}\, dw$$

Problems about computing line integrals using the CIF #0

Lemma (2.7) Let g be a rectifiable curve. Suppose that F_n and F are continuous on $\{g\}$ and that $\{F_n\}$ converges uniformly on $\{g\}$ to F. Then, $\lim_{n\to\infty} \int_g F_n = \int_g \lim_{n\to\infty} F_n = \int_g F$

Theorem 2.8 Let G be a region and let $f : G \to \mathbb{C}$ be analytic. Let $B(a,R) \subset G$. Then, f has a power series representation on $B(a,R)$, say

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n \tag{1}$$

where the coefficients $a_n = \dfrac{f^{(n)}(a)}{n!} = \dfrac{1}{2p\,i} \displaystyle\int_{|w-a|=r} \dfrac{f(w)}{(w-a)^{n+1}}\, dw$, for any choice $0 < r < R$.

Furthermore, the radius of convergence of the power series (1) is at least *R*.

Corollaries (Hypothesis: Let G be a region and let $f : G \to \mathbb{C}$ be analytic. Let $B(a,R) \subset G$.)

a) the radius of convergence of the power series (1) is equal to $\mathrm{dist}(a,\partial G)$, i.e., the distance (from a) to the nearest singularity of f

b) $f^{(n)}(a) = \frac{n!}{2p\,i} \int_{|w-a|=r} \frac{f(w)}{(w-a)^{n+1}}\, dw$

c) Cauchy's Estimate If $|f(z)| \le M$ on $B(a,R)$, then $|f^{(n)}(a)| \le \frac{n!M}{R^n}$.

d) f has a primitive on $B(a,R)$, namely $F(z) = \sum_{n=0}^{\infty} \frac{a_n}{n+1}(z-a)^{n+1}$

e) Proposition 2.15 Suppose g is a closed rectifiable curve in $B(a,R)$. Then, $\int_g f = 0$

Chapter 4.3

Division Algorthim

Definition: Let G be a region and let $f : G \to \mathbb{C}$ be analytic and let $f(a) = 0$. We say that f has a zero of order m (multiplicity m) at $z = a$ if

a) there exists $g \in \mathcal{A}\ (G)$ such that (i) $f(z) = (z-a)^m g(z)$ and (ii) $g(a) \neq 0$

or alternatively

b) (i) $f(a) = f'(a) = f''(a) = \cdots = f^{(m-1)}(a) = 0$ and $f^{(m)}(a) \neq 0$

Definition: entire function

Liouville's Theorem If f is a bounded entire function, then f is constant.

Fundamental Theorem of Algebra Every non-constant polynomial with complex coefficients has a root in $\mathbb{C}$

Corollary: Every polynomial with complex coefficients of degree n has exactly n roots (counted according to multiplicity)

<u>Identity Theorem</u>: Let G be a region. Let f be analytic on G. TFAE

a) $f \equiv 0$

b) there exists a point $a \in G$ such that $f^{(n)}(a) = 0,\ n = 0,1,2,3,\cdots$

c) $Z_f = \{z \in G : f(z) = 0\}$ has a limit point in G.

Corollaries:

a) Let $g, h \in \mathcal{A}(G)$ and $g(z) = h(z)$ for $z \in S \subset G$. If S has a limit point in G, then $g \equiv h$.

b) Let G be a region. Let f be analytic on G. Suppose that f is not identically 0 on G. If $a \in G$ and $f(a) = 0$, then there exists an integer m such that f has a zero at $z = a$ of multiplicity m.

c) Isolated Zeros. Let G be a region. Let f be analytic on G. Suppose that f is not identically 0 on G. If $a \in G$ and $f(a) = 0$, then there exists an $R > 0$ such that on $B(a, R) \setminus \{a\}$ we have $f(z) \neq 0$.

d) Let G be a region. Let f be analytic on G. Suppose that f is not identically 0 on G. Let K be a compact subset of G. Then, f has at most a finite number of zeros on K. Further, f has at most a countable number of zeros on G.

Maximum Modulus Theorem Let G be a region. Let f be analytic on G. If there exists a point $a \in G$ such that $|f(a)| \geq |f(z)|$ for all $z \in G$, then f is constant on G.

Extension Let G be a region. Let f be analytic on G. If there exists a point $a \in G$ and $r > 0$ such that $|f(a)| \geq |f(z)|$ for all $z \in B(a, r)$, then f is constant on G.

Chapter 4.4

Proposition: Let g be a closed rectifiable curve and let $a \notin \{g\}$. Then, $\frac{1}{2p i}\int_g \frac{dz}{z-a}$ is an integer.

Definition: Winding of g wrt to a (Index of g wrt to a) $n(g, a) = \frac{1}{2p i}\int_g \frac{dz}{z-a}$, for g a closed rectifiable curve and $a \notin \{g\}$.

Interpretation: $2p\ n(g, a)$ represents the total change in $\arg(g(t) - a)$ as $g(t)$ parametrizes the curve $\{g\}$, i.e., $n(g, a)$ represents the total number of times g winds around a.

Theorem Let g be a closed rectifiable curve. Then, $n(g, a)$ is constant on components of $\mathbb{C} \setminus \{g\}$. Also, $n(g, a) = 0$ on the unbounded component of $\mathbb{C} \setminus \{g\}$