Math 5320 | Take-Home Exam I | September 26, 2008

Due: October 1

1. (10 pts) It can be shown that circles and lines in the plane $\mathbb{C}$ project to circles on the sphere S and vice versa. In general, two distinct lines in the plane divide the plane into (at most) 4 pieces and three distinct lines in the plane divide the plane into (at most) 7 pieces – there may be fewer if some of the lines are parallel. A great circle on the sphere is a circle for which the center of the circle is the center of the sphere (or, equivalently, a circle which divides the sphere into two congruent pieces (half spheres)).

   a. Into how many pieces (at most) do two distinct great circles divide the sphere? Into how many pieces (at most) do three distinct great circles divide the sphere?

   b. Into how many pieces (at most) do four distinct lines divide the plane? Into how many pieces (at most) do four distinct great circles divide the sphere?

2. (10 pts) Sketch each of the following sets T in $\mathbb{C}$:

   a. Let $S = \{z : \operatorname{Re} z < 0 \text{ and } \operatorname{Im} z > 0 \text{ and } |z| < 1\}$. Then,

   $$T = \{w : w = \frac{1}{z}, z \in S\}$$

   b. $T = \{z : \operatorname{Im} z^2 > 4\}$

3. (10 pts) For each of the following cases, give an example of a metric space (X, d), where $X \subset \mathbb{C}$ and d is the inherited metric such that

   a. X has exactly three limit points.

   b. there exists a sequence $\{z_n\} \subset X$ such that $\{z_n\}$ is Cauchy, but $\{z_n\}$ does not converge.

4. (10 pts) Prove that if $|z| < 1$, then $\operatorname{Re} \frac{1+z}{1-z} > 0$.

5. (10 pts) Let $M = \left\{ \begin{bmatrix} a & b \\ -b & a \end{bmatrix} : a, b \in \mathbb{R} \right\}$ and let $\oplus, \odot$ denote matrix addition and matrix multiplication, resp.. Show that $(\mathbb{C}, +, \bullet)$ and $(M, \oplus, \odot)$ are isomorphic (fields).