MATH 5320 Exam II October 25, 2004

TakeHome - Due Nov 8

Answer the problems on separate paper. You do <u>not</u> need to rewrite the problem statements on your answer sheets. Work carefully. Do your own work. **<u>Show all relevant supporting steps!</u>**

1. (15) Discuss the convergence of five of the following series, i.e., determine where the series converge, where the series converge absolutely and where the series converge uniformly.

a. $\sum_{n=0}^{\infty}\left(\frac{z}{z+1}\right)^n$ b. $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{z+n}$ c. $\sum_{n=1}^{\infty}\frac{z^n}{1-z^n}$

d. $\sum_{n=1}^{\infty}\frac{1}{z^2+n^2}$ e. $\sum_{n=1}^{\infty}n^{-z}$ f. $\sum_{n=0}^{\infty}e^{-nz^2}$

2. (9) Suppose that the radius of convergence of $\sum_{n=0}^{\infty}a_n z^n$ is r, where $0 < r < \infty$. Find the radius of convergence of each of the following series:

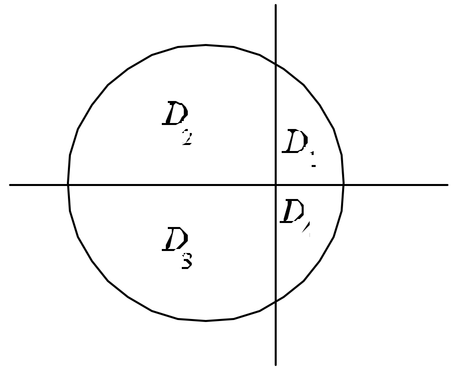
a. $\sum_{n=0}^{\infty}a_n z^{2n}$ b. $\sum_{n=0}^{\infty}a_{2n} z^n$ c. $\sum_{n=0}^{\infty}a_n{}^2 z^n$

3. (4) Let $G_1 = \{z : 0 < \operatorname{Re} z < p, \operatorname{Im} z > 0\}$. Let $f(z) = \cos z$. Find the conformal image of G_1 under f. Explicitly show that the images of the curves $m_x = \{z = x + iy : y > 0\}$ and $n_y = \{z = x + iy : 0 < x < p\}$ under f intersect orthogonally in $f(G_1)$.

4. (4) Let $G_2 = \{z : |z - \tfrac{1}{2}| < \tfrac{1}{2}\}$ and $G_3 = \{z : |\operatorname{Im} z| < 1\}$. Find a one-to-one conformal mapping f which maps G_2 onto G_3. Hint: Find a one-to-one conformal mapping f_1 which maps G_2 onto *RHP* (the open right half-plane). Then, find one-to-one conformal mapping f_2 which maps *RHP* onto G_3.

5. (10) The lines, $x = 1/2$ and $y = 0$ divide the interior of the unit circle into 4 subregions D_1, D_2, D_3 and D_4. See figure to the right. Let $w = \dfrac{z}{z+1}$. Find the images E_j of each subregion D_j under w, i.e., find

$$E_j = w(D_j), \quad j = 1,2,3,4.$$



6. (4) Let $G_4 = \{z = re^{iq} : 0 < r < 1, 0 < q < p/2\}$ and let $G_5 = \{z = re^{iq} : 0 < r < \infty, 0 < q < p/2\}$, i.e., G_5 is the first quadrant (the interior thereof) and G_4 is the intersection of the first quadrant with the unit disk centered at the origin (the interior thereof). Find a one-to-one conformal mapping f which maps G_4 onto G_5.

7. (4) Let $u(x, y) = x^3 - 3xy^2 + 2e^y \cos x + 1$. Show that u is harmonic on *RHP* (the open right half-plane) and find a harmonic conjugate v such that $f = u + iv$ is analytic on *RHP*.