MATH 5320 Exam I October 6, 1997

TakeHome

Answer the problems on separate paper. You do <u>not</u> need to rewrite the problem statements on your answer sheets. Work carefully. Do your own work. **<u>Show all relevant supporting steps!</u>**

1. Prove: $\left|\dfrac{a - b}{1 - \bar{a}b}\right| \leq 1$ if $|a| \leq 1$, $|b| \leq 1$ (but not both $|a| = 1$ and $|b| = 1$) with equality if and only if $|a| = 1$ or $|b| = 1$.

2. Prove: $Re\, z < 0, z \neq -1 \Rightarrow \left|\dfrac{z-1}{z+1}\right| > 1$.

3. Find the radius of convergence of $\sum_{n=1}^{\infty} n^{\log n} z^n$.

4. Suppose $\left|\dfrac{z-1}{z+1}\right| = \dfrac{1}{2}$. Characterize z.

5. Let $G \subset \mathbb{C}$, G open, and let $f : G \to \mathbb{C}$, $f(z) = u(x,y) + i\, v(x,y)$, $z = x + iy$. Prove that if f is continuous on G, then u and v are continuous on G.

6. Find the domain of absolute convergence of $\sum_{n=1}^{\infty} e^{-2z \log n}$.

7. Find the domain of absolute convergence of $\sum_{n=1}^{\infty} \left(\dfrac{z}{1+z}\right)^n$.

8. Let $z_0 = 1$, $z_1 = i$ and $z_n = \dfrac{z_{n-1} + z_{n-2}}{2}$, $n > 1$. If $\{z_n\}$ converges, to what does it converge?

9. Suppose that D is dense in (X,d). Suppose (Ω,ρ) is a complete metric space. Suppose that $f : (D,d) \to (\Omega,\rho)$ is uniformly continuous on D. Show that there exists a function $g : (X,d) \to (\Omega,\rho)$ such that g is uniformly continuous on X and $g(x) = f(x)$ for all $x \in D$.

10. Is the set $A = \{ z : |z| - Im z \leq 1 \}$ bounded?