Review Summary

Chapters 9.1 - 11.6

Chapter 9

A. Convergence of Functions
   1. Sequence of Functions $\{ f_n \}$
      a. Pointwise Convergence on [a,b]
      b. Uniform Convergence on [a,b]
   2. Series of Functions $\sum u_n$
      a. Pointwise Convergence on [a,b]
      b. Uniform Convergence on [a,b]
   3. Examples of Sequences (Series) which Converge Pointwise but Not Uniformly
   4. Examples of Sequences (Series) which Converge Uniformly

B. Dini's Theorem

C. Weierstrass M-Test for Uniform Convergence

D. Consequences of Uniform Convergence

Sequence of Functions $\{ f_n \}$	Series of Functions $\sum u_n$	Power Series $\sum a_n x^n$
f_n cont. on [a,b], $f_n \to f$ unif. on [a,b] $\Big\}$ f cont. on [a,b]	u_n cont. on [a,b], $\sum u_n \to f$ unif. on [a,b] $\Big\}$ f cont. on [a,b]	$\sum a_n x^n$ converges at x_0, $x_0 \neq 0$ $\Big\}$ $\sum a_n x^n$ cont. on $[-x_1,x_1]$ for $0 < x_1 < \lvert x_0 \rvert$
$f_n \in \Re[a,b]$, $f_n \to f$ unif. on [a,b] $\Big\}$ $f \in \Re[a,b]$	$u_n \in \Re[a,b]$, $\sum u_n \to f$ unif. on [a,b] $\Big\}$ $f \in \Re[a,b]$	$\sum a_n x^n$ converges at x_0, $x_0 \neq 0$ $\Big\}$ $\sum a_n x^n \in \Re[-x_1,x_1]$ for $0 < x_1 < \lvert x_0 \rvert$
$f_n \in \Re[a,b]$, $f_n \to f$ unif. on [a,b] $\Big\}$ $\lim \int f_n = \int \lim f_n$	$u_n \in \Re[a,b]$, $\sum u_n \to f$ unif. on [a,b] $\Big\}$ $\int \sum u_n = \sum \int u_n$	$\sum a_n x^n$ converges at x_0, $x_0 \neq 0$ $\Big\}$ $\int_0^{x_1} \sum a_n x^n \, dx = \sum \frac{a_n}{n+1} x_1^{n+1}$

Sequence of Functions $\{ f_n \}$	Series of Functions $\sum u_n$	Power Series $\sum a_n x^n$
f_n cont. on [a,b] f_n' cont. on [a,b] $f_n \longrightarrow f$ on [a,b] $f_n' \longrightarrow g$ unif. on [a,b] $\Bigg\}$ f' exists on [a,b] and $f' = g$	u_n cont. on [a,b] u_n' cont. on [a,b] $\sum u_n \longrightarrow f$ on [a,b] $\sum u_n' \longrightarrow g$ unif. on [a,b] $\Bigg\}$ f' exists on [a,b] and $f' = g$	$\sum a_n x^n$ converges at x_0, $x_0 \neq 0$ $\Bigg\}$ $\frac{d}{dx} \sum a_n x^n$ exists on $[-x_1, x_1]$ for $0 < x_1 < \lvert x_0 \rvert$ and $\frac{d}{dx} \sum a_n x^n = \sum n a_n x^{n-1}$

E. Definition of Abel Summability
1. Examples of Series which are Abel Summable
2. Examples of Series which are Not Abel Summable

F. Theorems Whose Proofs You Should Know
1. Theorem 9.3B
2. Theorem 9.3G

G. Representative Problems
1. 9.1: 2-5
2. 9.2: 1, 4-5, 8
3. 9.3: 1-2, 4
4. 9.4: 1-4
5. 9.5: 1-3
6. 9.6: 1

Chapter 10.

A. Metric Space $\mathcal{C}[a,b]$
1. Definition of sup norm $\| f \|$
2. Definition of metric ρ: $\rho(f, g) = \| f - g \|$
3. Theorem 10.1D: $\{ f_n \} \longrightarrow f$ in metric ρ if and only if $\{ f_n \} \longrightarrow f$ uniformly on [a,b]
4. $\mathcal{C}[a,b]$ is complete.

B. Weierstrass Approximation Theorem

C. Picard's Existence Theorem

D. Arzela-Ascoli Theorem
1. Definition of Equicontinuity
2. Examples of Families $\mathfrak{F}$ which are Equicontinuous
3. Examples of Families $\mathfrak{F}$ which are Not Equicontinuous

E. Representative Problems
1. 10.1: 1
2. 10.4: 1-4

Chapter 11.

A. Length of Open Set / Closed Sets
1. Definition of Length of an Open Set
2. Theorem 11.1B: Length Estimate of Countable Union of Open Sets $|\bigcup G_n| \le \sum |G_n|$
3. $|G_1| + |G_2| = |G_1 \cup G_2| + |G_1 \cap G_2|$
4. Definition of Length of Closed Set
5. Computation of Lengths of Example Open Sets and Closed Sets

B. Measurable Sets ($E \subset$ [a,b])
1. Definition of Outer Measure $\overline{m}E$ and Inner Measure $\underline{m}E$
2. Definition of a Measurable Set
3. $\underline{m}E \le \overline{m}E$
4. E' is measurable whenever E is measurable.
   a. $mE' =$ (b-a) - mE
5. Open Sets are Measurable
   a. $mG = |G|$
6. Closed Sets are Measurable
   b. $mF = |F|$
7. $mE_1 + mE_2 = m(E_1 \cup E_2) + m(E_1 \cap E_2)$
8. $E_1 \setminus E_2$ is measurable whenever E_1 and E_2 are measurable.
9. Theorem 11.3E: Countable Union of Pair-wise Disjoint Measurable Sets is Again Measurable
   a. $m(\bigcup E_n) = \sum m(E_n)$
10. Theorem 11.3H: Countable Union of Measurable Sets is Again Measurable
   a. $m(\bigcup E_n) \le \sum m(E_n)$
   b. Countable Intersection of Measurable Sets is Again Measurable

C. Measurable Functions ($f : [a,b] \longrightarrow \mathbb{R}$, $g : [a,b] \longrightarrow \mathbb{R}$)

1. Definition of Measurable Function
2. Theorem 11.4B: Three Equivalent Formulations to Definition of Measurable Function
3. Theorem 11.4D: If $f = g$ a.e. on [a,b] and if f is measurable, then g is measurable
4. Algebra of Measurable Functions ($f + c$, cf, $f + g$, $f - g$, f^2, fg, $1/g$, f/g)
5. Theorem 11.4H: If a sequence $\{ f_n \}$ of measurable functions converges to f on [a,b], then f is again measurable.
6. Examples of Measurable Functions
7. Examples of Functions which are Not Measurable

D. Lebegue Integration

1. Parallel Implementation to Riemann Integration with Partition of [a,b] into Subintervals Replaced by Partition of [a,b] into Measurable Sets.
2. Definition of Measurable Partition of [a,b]
3. Definition of Upper and Lower Sums for a Bounded Function on [a,b]
4. Definition of Upper and Lower Lebesgue Integrals for a Bounded Function on [a,b]
5. $\mathfrak{R}\underline{\int_a^b} f \le \mathfrak{L}\underline{\int_a^b} f \le \mathfrak{L}\overline{\int_a^b} f \le \mathfrak{R}\overline{\int_a^b} f$
6. Definition of Lebesgue Integral for a Bounded Function on [a,b]
7. Theorem 11.5G: $\mathfrak{R}[a,b] \subset \mathfrak{L}[a,b]$.
   a. If $f \in \mathfrak{R}[a,b]$, then $\mathfrak{R}\int_a^b f = \mathfrak{L}\int_a^b f$.
8. Theorem 11.5I: If f is bounded measurable function on [a,b], then $f \in \mathfrak{L}[a,b]$.

Properties of Lebegsue Integration

Riemann Integration $f, g \in \mathfrak{R}[a,b] \Leftrightarrow f, g$ bdd, cont. a.e. on [a,b]		Lebesgue Integration f, g bdd, measurable on [a,b]	
7.4A	$\int_a^b f = \int_a^c f + \int_c^b f$	11.6A	$\int_a^b f = \int_a^c f + \int_c^b f$
7.4B	$\int_a^b \lambda f = \lambda \int_a^b f$	11.6B	$\int_a^b \lambda f = \lambda \int_a^b f$
7.4C	$\int_a^b f + g = \int_a^b f + \int_a^b g$	11.6C	$\int_a^b f + g = \int_a^b f + \int_a^b g$
		11.6D	$f = g$ a.e. $\Longrightarrow \int_a^b f = \int_a^b g$
7.4D	$f \ge 0$ a.e. $\Longrightarrow \int_a^b f \ge 0$	11.6E	$f \ge 0$ a.e. $\Longrightarrow \int_a^b f \ge 0$

Riemann Integration $f, g \in \mathcal{R}[a,b] \Leftrightarrow f, g$ bdd, cont. a.e. on [a,b]		Lebesgue Integration f, g bdd, measurable on [a,b]	
7.4E	$f \le g$ a.e. $\Longrightarrow \int_a^b f \le \int_a^b g$	11.6F	$f \le g$ a.e. $\Longrightarrow \int_a^b f \le \int_a^b g$
7.4F	$\left\|\int_a^b f\right\| \le \int_a^b \|f\|$	11.6G	$\left\|\int_a^b f\right\| \le \int_a^b \|f\|$
7.4 Ex. 8	$f \in \mathcal{C}[a,b]$ and $f(x) \ge 0$ and $\int_a^b f = 0$ $\Longrightarrow f(x) = 0$ for all $x \in [a,b]$	11.6M	f bdd, measurable on [a,b] and $f(x) \ge 0$ and $\int_a^b f = 0$ $\Longrightarrow f(x) = 0$ on [a,b] a.e.

E. Theorem 11.6N: If f is bounded on [a,b] and $f \in \mathcal{L}[a,b]$, then f is measurable.

F. Theorems Whose Proofs You Should Know

1. Theorem 11.2C
2. Corollary 11.3B
3. Corollary 11.3E
4. Theorem 11.4B
5. Theorem 11.5G
6. Theorem 11.6B, 11.6C
7. Theorem 11.6E

G. Representative Problems

1. 11.1: 2-5
2. 11.2: 2-3
3. 11.3: 2-4
4. 11.4: 1, 3
5. 11.5: 2-3
6. 11.6: 1, 3, 5