MATH 3360 Final Exam May 9, 1995

Show your work for each problem. You do <u>not</u> need to rewrite the statements of the problems on your answer sheets. Each problem is worth 10 points.

Section I. Do any four problems.

1. Which elements of $\mathbf{Z}_{15}$ are zero divisors?

2. Let $R = M(2,\mathbf{Z})$. Let $S = \{ \begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix} : a,b \in \mathbf{Z} \}$. Determine whether S is a subring of R.

3. For each of the following conditions give an example which satisfies the condition if such an example exists; if no example exists, explain why.

   a) A finite commutative ring with unity which is not an integral domain.
   b) A finite integral domain which is not a field.
   c) A finite field which has zero divisors.

4. Let E and F be fields such that $E \approx F$. Let θ be an isomorphism which maps E to F. Suppose $a \in E$ and $a \neq 0$. Prove that $\theta(a^{-1}) = \theta(a)^{-1}$.

5. Let R denote the field $\{a + b\sqrt{2} : a,b \in \mathbf{Q} \}$ and let S denote the field $\{a + b\sqrt{3} : a,b \in \mathbf{Q} \}$. Prove the mapping $\theta : R \to S$ defined by $\theta(a + b\sqrt{2}) = a + b\sqrt{3}$ is not a ring isomorphism.

Section II. Do any one problem.

1. Let $O = \{ f : f = \frac{a}{b}$ where both a,b are positive odd integers $\}$. Define 'addition' for $f,g \in O$ by $f + g = fg$ where fg is standard multiplication. Define 'multiplication' for $f,g \in O$ by $f \cdot g = 1$. Determine whether O with its 'addition' and 'multiplication' is a ring.

2. Let $O = \{ f : f = \frac{a}{b}$ where both a,b are positive odd integer$\}$. Define 'addition' for $f,g \in O$ by $f + g = fg$ where fg is standard multiplication. Define 'multiplication' for $f,g \in O$ by $f \cdot g = f^2g^2$. Determine whether O with its 'addition' and 'multiplication' is a ring.

Section III. Do any two problems.

1. Let D be an ordered integral domain. Let $a,b,c \in D$ and suppose that both $ac > bc$ and $c > 0$. Prove that $a > b$.

2. Let D be a well-ordered integral domain. Let $a \in D$ and suppose that both $a \neq 0$ and $a \neq e$, where e is the unity of D. Prove that $a^2 > a$.

3. Let D be an ordered integral domain. Suppose that E is a subring of D. Prove that E is also an ordered integral domain.

Section IV. Do any five problems.

1. Let R be $M(2,\mathbf{Z})$. Let $B = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$. Consider the mappings λ,μ as mappings from the additive group R to the additive group R where λ is given by $\lambda(A) = BA$ for $A \in R$ and μ is given by $\mu(A) = AB$ for $A \in R$. Find $\ker(\lambda)$ and $\ker(\mu)$.

2. Let G,H be groups and let θ be a group homomorphism, $\theta : G \to H$. Suppose that A is a subgroup of G. Prove that $\theta(A)$ is a subgroup of H.

3. Let G,H be groups and let θ be a group homomorphism, $\theta : G \to H$. Let $a \in G$. Prove that $o(\theta(a)) \mid o(a)$.

4. Prove or disprove: Suppose $N \triangleleft G$. Then, $gng^{-1} = n$ for all $n \in N$ and for all $g \in G$.

5. Suppose $N \triangleleft G$. Prove that if $[G : N]$ is prime, then G/N is cyclic.

6. Prove that if G is a simple Abelian group, then $G \approx \mathbf{Z}_p$ for some prime p.