Chapter 5 Power Series Solution of ODEs

5 Introduction

Notes Proofread by Yunting Gao and corrections made on 04/05/2021

5.1.1 Review of Power Series

Convergence of Sequences and Series

I) We say a sequence of numbers $\{S_n\}_{n=1}^{\infty}$ converges to a number $L \in \mathbb{R}$ if $\lim\limits_{n\to\infty} S_n = L$, i.e., the limit exists and the limit is the number L (which we call the limit of the sequence).

   A) If a sequence is increasing and bounded above, then it converges.

   B) If a sequence is decreasing and bounded below, then it converges.

   C) If $S_k = f(k)$ where f is differentiable then $\lim\limits_{n\to\infty} S_n = \lim\limits_{x\to\infty} f(x)$. This useful fact allows us to find limits of some sequences using tools from calculus.

II) We say a series $\sum\limits_{k=1}^{\infty} a_k$ converges if the sequence of partial sums, $S_n = \sum\limits_{k=1}^{n} a_k$ converges.

   A) A geometric series $\sum\limits_{k=0}^{\infty} ar^k$ either $\begin{cases} \text{converges to } \dfrac{a}{1-r}, & \text{if } |r| < 1 \\ \text{diverges} & \text{if } |r| \geq 1 \end{cases}$

   B) If a series with nonnegative terms has all partial sums bounded, then the series converges by the bounded monotone convergence theorem.

   C) (Divergence test) If $\lim\limits_{k\to\infty} a_k \neq 0$ then the series $\sum\limits_{k=1}^{n} a_k$ diverges.

   D) (Integral Test) If $a_k = f(k)$ where f is continuous and eventually positive and decreasing for $x > a > 0$, then

$$\int_a^{\infty} f(x)\,dx \quad \text{and} \quad \sum_{k=1}^{\infty} a_k \ \text{ either both converge or diverge.}$$

E) (p-test) The series $\displaystyle\sum_{k=0}^{\infty} \frac{1}{k^p} = \begin{cases} \text{converges}, & p > 1 \\ \text{diverges} & p \leq 1 \end{cases}$

F) (Direct Comparison) If $0 < d_k \leq a_k \leq c_k$, then

(i) If $\displaystyle\sum_{k=1}^{\infty} d_k$ diverges $\Rightarrow \displaystyle\sum_{k=1}^{\infty} a_k$ diverges.

(ii) If $\displaystyle\sum_{k=1}^{\infty} c_k$ converges $\Rightarrow \displaystyle\sum_{k=1}^{\infty} a_k$ converges.

G) (Limit Comparison) If $0 < b_k,\ a_k$ and $L = \displaystyle\lim_{k\to\infty} \frac{a_k}{b_k}$ with $0 < L < \infty$, then the series

$$\sum_{k=1}^{\infty} a_k \text{ and } \sum_{k=1}^{\infty} b_k \text{ either both converge or both diverge.}$$

H) (Ratio test) Let $L = \displaystyle\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right|$ then $\begin{cases} L < 1 \quad \Rightarrow \quad \text{series } \displaystyle\sum_{k=1}^{\infty} a_k \text{ converges} \\ L > 1 \quad \Rightarrow \quad \text{series } \displaystyle\sum_{k=1}^{\infty} a_k \text{ diverges} \\ L = 1 \quad \text{the test fails} \end{cases}$.

I) (Root test) Let $L = \displaystyle\lim_{k\to\infty} \sqrt[k]{|a_k|}$ then we have the same result as above.

J) (Alternating Series test) Assume $\{a_k\}_{k=1}^{\infty}$ are positive and (eventually) **decreasing to zero**, then the alternating series $\displaystyle\sum_{k=1}^{\infty} (-1)^k a_k$ converges. For a convergent alternating series we have $|S - S_n| \leq a_{k+1}$.

K) If a series converges absolutely (i.e., $\displaystyle\sum_{k=1}^{\infty} |a_k|$ converges) then $\displaystyle\sum_{k=1}^{\infty} a_k$ converges. The converse is not true as shown by the Harmonic series:

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \quad \text{converges but} \quad \sum_{n=1}^{\infty} \frac{1}{n} \quad \text{diverges.}$$

Taylor and MacLaurin Series

An expression in the form $\displaystyle\sum_{n=0}^{\infty} a_n (x-a)^n$ is called a *Power Series* centered at $x = a$. An *Analytic* function $f(x)$ is a function which possesses a convergent power series, i.e.,

$f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n$ where the series converges absolutely on an interval $|x-a| < R$. The number R is called the radius of convergence. If f is analytic then the coefficients a_n must be given by the so called Taylor Coefficients and the series is called a Taylor series. Namely we have

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

When $a = 0$ we call the Taylor series a *MacLaurin Series*. Such series tend to be a bit simpler and so that is what we focus on for the most part in the following discussion.

Smooth Non-Analytic Functions

One might wonder whether there are smooth functions, i.e., functions for which all derivatives of all order exist, which do not posses representation as a convergent power series. The answer is yes! Here is an example.

$$f(x) = \begin{cases} e^{1/x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}.$$

has all derivatives zero there. Consequently, the Taylor series of f(x) about x = 0 is identically zero. However, f(x) is not equal to the zero function, and so it is not equal to its Taylor series around the origin.

Taylor Polynomial

Since in practice one cannot usually sum an infinite number of terms or find a closed form for the sum it is often useful to consider truncating the infinite sum of a power series which produces a polynomial. Sometimes this polynomial can be used as a accurate approximation of the function.

Theorem 5.1 (Taylor's Theorem with Remainder)**.** *Let $N > 0$ be an integer and f be a function which is N times differentiable at a point $x = a$. Then we have*

$$f(x) = \sum_{n=0}^{N} \frac{f^{(n)}(a)}{n!}(x-a)^n + R_N(x)$$

where

$$R_N(x) = \frac{1}{n!} \int_a^x (x-t)^n f^{(N+1)}(t)\, dt.$$

When $a = 0$ the Taylor series is called a *MacLaurin Series*.

Examples of Common MacLaurin Series

$f(x)$	$\sum_{n=0}^{\infty} a_n x^n$
e^x	$\sum_{n=0}^{\infty} \frac{x^n}{n!}$
$\cos(x)$	$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$
$\sin(x)$	$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$
$\frac{1}{1-x}$	$\sum_{n=0}^{\infty} x^n$
$\ln(1+x)$	$\sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{(n+1)}$
$(1+x)^p$	$1 + px + \frac{p(p-1)}{2!}x^2 + \frac{p(p-1)(p-2)}{3!}x^3 + \cdots + \binom{p}{n} x^n$

Adding Power Series and Shifting Indices

Given a power series $f(x) = \sum_{n=0}^{\infty} a_n x^n$, at least for x within the radius of convergence, we can differentiate the series to obtain $f'(x) = \sum_{n=0}^{\infty} a_n n x^{n-1}$. Furthermore we can repeat this as many times as we like and we still obtain a convergent power series. So, for example, $f''(x) = \sum_{n=0}^{\infty} a_n n(n-1)x^{n-2}$. Suppose now we want to form an expression like

$$f''(x) + xf'(x) - f(x)$$

into a single power series, i.e., we want to add the series together. Then we would write

$$\sum_{n=0}^{\infty} a_n n(n-1)x^{n-2} + x\sum_{n=0}^{\infty} a_n n x^{n-1} - \sum_{n=0}^{\infty} a_n x^n.$$

This presents a problem since you cannot combine apples and oranges. By this we mean that the powers of x in each sum are different and you cannot combine directly x^n with x^{n-2} for example. This problem can be easily fixed by *Shifting Indices*. Here is an example of what I mean. In the series $\sum_{n=0}^{\infty} a_n x^n$ we can shift the index n inside the sum down by two provided we shift the indices in the summation up by two to obtain exactly the same sum, i.e.,

$$\sum_{n=0}^{\infty} a_n x^n = \sum_{n=2}^{\infty} a_{n-2} x^{n-2}.$$

Similarly

$$x\sum_{n=0}^{\infty} a_n n x^{n-1} = \sum_{n=0}^{\infty} a_n n x^n = \sum_{n=2}^{\infty} a_{n-2}(n-2)x^{n-2}.$$

So we can write

$$\begin{aligned}
f''(x) + xf'(x) - f(x) &= \sum_{n=0}^{\infty} a_n n(n-1)x^{n-2} + x\sum_{n=0}^{\infty} a_n n x^{n-1} - \sum_{n=0}^{\infty} a_n x^n \\
&= \sum_{n=0}^{\infty} a_n n(n-1)x^{n-2} + \sum_{n=2}^{\infty} a_{n-2}(n-2)x^{n-2} - \sum_{n=2}^{\infty} a_{n-2}x^{n-2} \\
&= \sum_{n=0}^{\infty} a_n n(n-1)x^{n-2} + \sum_{n=2}^{\infty} a_{n-2}\big[(n-2) - 1\big]x^{n-2} \\
&= \sum_{n=0}^{\infty} a_n n(n-1)x^{n-2} + \sum_{n=2}^{\infty} a_{n-2}(n-3)x^{n-2} \\
&= 0a_0x^{-2} + 0a_1x^{-1} + \sum_{n=2}^{\infty} a_n n(n-1)x^{n-2} + \sum_{n=2}^{\infty} a_{n-2}(n-3)x^{n-2} \\
&= \sum_{n=2}^{\infty} \big[n(n-1)a_n + (n-3)a_{n-2}\big]x^{n-2}
\end{aligned}$$

5.1.2 Power Series Solutions of ODEs

In this section we consider the problem of solving an ordinary differential equation of the form

$$P(x)y'' + Q(x)y' + R(x)y = 0 \tag{1}$$

in the form of a power series $f(x) = \sum_{n=0}^{\infty} a_n(x-a)^n$. In this discussion we will only consider the simpler case in which $a = 0$ so we are looking for a solution as a MacLaurin Series. So, in particular, we seek solution near $x = 0$.

In order that an equation in the form (1) have a solution which is analytic (has a convergent power series) some assumptions must be made. First we must assume that $P(x)$, $Q(x)$ and $R(x)$ are analytic functions. In addition we must assume that $P(x)$ is not zero at or near $x = 0$. More general cases are considered in the next few sections of Chapter 5 but we will not have time to consider these cases.

Let us consider a very simple example that we solved back in Chapter 2 (a first order linear example).

Example 5.1. Find the general solution of $y' - y = 0$ in the form $y = \sum_{n=0}^{\infty} a_n x^n$.

First we compute $y' = \sum_{n=0}^{\infty} a_n n x^{n-1}$ and then we substitute these series into the differential equation and try to find coefficients a_n so that the resulting equation is satisfied. We have

$$\begin{aligned} 0 = y' - y &= \sum_{n=0}^{\infty} a_n n x^{n-1} - \sum_{n=0}^{\infty} a_n x^n \\ &= \sum_{n=0}^{\infty} a_n n x^{n-1} - \sum_{n=1}^{\infty} a_{n-1} x^{n-1} \\ &= 0a_0 x^{-1} + \sum_{n=1}^{\infty} a_n n x^{n-1} - \sum_{n=1}^{\infty} a_{n-1} x^{n-1} \\ &= 0a_0 x^{-1} + \sum_{n=1}^{\infty} \left[a_n n - a_{n-1}\right] x^{n-1}. \end{aligned}$$

The first term is zero for all nonzero values of x so we see that a_0 can be any real number, i.e. it is an arbitrary constant.

Now, in order that the remaining equation

$$\sum_{n=1}^{\infty} \left[a_n n - a_{n-1}\right] x^{n-1} = 0$$

holds for all $x \neq 0$ we would need

$$a_n n - a_{n-1} = 0, \quad n = 1, 2, \cdots .$$

This is the same as

$$a_n = \frac{a_{n-1}}{n}, \quad n = 1, 2, \cdots \quad \text{(Recursion Formula).}$$

The Recursion Formula can be used to successively obtain the terms a_n in terms of a_0 as follows.

$$\boldsymbol{n=1}, \quad a_1 = \frac{a_0}{1} \qquad \boldsymbol{n=2}, \quad a_2 = \frac{a_1}{2} = \frac{a_0}{2 \cdot 1}$$

$$\boldsymbol{n=3}, \quad a_3 = \frac{a_2}{3} = \frac{a_0}{3!} \qquad \boldsymbol{n=4}, \quad a_4 = \frac{a_3}{4} = \frac{a_0}{4!}$$

It is easy to see the pattern and we can extrapolate the above to conclude that

$$a_n = \frac{a_0}{n!} \quad \text{for all} \quad n = 1, 2, 3, \cdots .$$

So we have

$$y = a_0 + \sum_{n=1}^{\infty} a_n x^n = \sum_{n=0}^{\infty} \frac{a_0}{n!} x^n = a_0 \sum_{n=0}^{\infty} \frac{x^n}{n!} = a_0 e^x.$$

From the first entry in our table of power series examples we see that $y = a_0 e^x$.

Next we consider a slightly more complicated example which, once again, we could easily solve using methods developed in Chapter 3.

Example 5.2. Find the general solution of $y'' + y = 0$ in the form $y = \sum_{n=0}^{\infty} a_n x^n$.

First we compute $y' = \sum_{n=0}^{\infty} a_n n x^{n-1}$ and $y'' = \sum_{n=0}^{\infty} a_n n(n-1) x^{n-2}$. Then we substitute these series into the differential equation and try to find the coefficients a_n so that the resulting equation is satisfied. We have

$$\begin{aligned}
0 = y'' + y &= \sum_{n=0}^{\infty} a_n n(n-1) x^{n-2} + \sum_{n=0}^{\infty} a_n x^n \\
&= \sum_{n=0}^{\infty} a_n n(n-1) x^{n-2} + \sum_{n=2}^{\infty} a_{n-2} x^{n-2} \\
&= 0 a_0 x^{-2} + 0 a_1 x^{-1} + \sum_{n=2}^{\infty} a_n n(n-1) x^{n-2} + \sum_{n=2}^{\infty} a_{n-2} x^{n-2} \\
&= 0 a_0 x^{-2} + 0 a_1 x^{-1} + \sum_{n=2}^{\infty} \big[a_n n(n-1) + a_{n-2}\big] x^{n-2}.
\end{aligned}$$

The first two terms are zero for all nonzero values of x so we see that a_0 and a_1 can be any real numbers, i.e. they are arbitrary constants.

Now, in order that the remaining equation

$$\sum_{n=1}^{\infty} \big[a_n n(n-1) + a_{n-2}\big] x^{n-2} = 0$$

holds for all $x \neq 0$ we would need

$$a_n n(n-1) + a_{n-2} = 0, \quad n = 2, 3, \cdots .$$

This is the same as

$$a_n = \frac{-a_{n-2}}{n(n-1)}, \quad n = 2, 3, \cdots \quad \text{(Recursion Formula)}.$$

The Recursion Formula can be used to successively obtain the terms a_n in terms of a_0

and a_1 as follows. In this present case we see that the

$$\boldsymbol{n=2}, \qquad a_2 = \frac{-a_0}{2\cdot 1} \qquad\qquad \boldsymbol{n=3}, \qquad a_3 = \frac{-a_1}{3\cdot 2}$$

$$= \frac{-a_0}{2!} \qquad\qquad\qquad = \frac{-a_1}{3!}$$

$$\boldsymbol{n=4}, \qquad a_4 = \frac{-a_2}{4\cdot 3} \qquad\qquad \boldsymbol{n=5}, \qquad a_5 = \frac{-a_3}{5\cdot 4}$$

$$= \frac{(-1)^2 a_0}{4!} \qquad\qquad\qquad = \frac{(-1)^2 a_1}{5!}$$

$$\boldsymbol{n=6}, \qquad a_6 = \frac{-a_4}{6\cdot 5} \qquad\qquad \boldsymbol{n=7}, \qquad a_7 = \frac{-a_5}{7\cdot 6}$$

$$= \frac{(-1)^3 a_0}{6!} \qquad\qquad\qquad = \frac{(-1)^3 a_1}{7!}$$

It is easy to see the two patterns that evolve, one for the even coefficients in terms of a_0 and the odd coefficients in terms of a_1. Every even integer can be written as $n = 2k$ for $k = 0, 1, 2, \cdots$ and every odd integer can be written as $n = 2k+1$ for $k = 1, 2, \cdots$. Therefore we can write the even a_n terms for $n \geq 2$ as

$$a_{2k} = \frac{(-1)^k a_0}{(2k)!} \quad \text{for all} \quad k = 1, 2, 3, \cdots ,$$

and we can write the odd a_n terms for $n \geq 3$ as

$$a_{2k+1} = \frac{(-1)^k a_1}{(2k+1)!} \quad \text{for all} \quad k = 1, 2, 3, \cdots .$$

So we have

$$y = \sum_{n=0}^{\infty} a_n x^n = a_0 + \sum_{k=1}^{\infty} a_{2k} x^{2k} + a_1 x + \sum_{k=1}^{\infty} a_{2k+1} x^{2k+1}.$$

So we have

$$y = a_0 + a_0 \sum_{k=1}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k} + a_1 x + a_1 \sum_{k=1}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}.$$

From the second and third entries in our table of power series examples we see that

$y = a_0 \cos(x) + a_1 \sin(x)$.

Next lets consider an example with non-constant coefficients and an initial value problem. Once again this is an example that we could have solved back in Chapter 2 since it is first order linear.

Example 5.3. Solve the initial value problem $y' - 2xy = 0$ with $y(0) = 1$ in the form of a series $y = \sum_{n=0}^{\infty} a_n x^n$.

First we compute $y' = \sum_{n=0}^{\infty} a_n n x^{n-1}$ and then we substitute these series into the differential equation and try to find coefficients a_n so that the resulting equation is satisfied. We have

$$\begin{aligned}
0 = y' - 2xy &= \sum_{n=0}^{\infty} a_n n x^{n-1} - 2\sum_{n=0}^{\infty} a_n x^{n+1} \\
&= \sum_{n=0}^{\infty} a_n n x^{n-1} - \sum_{n=2}^{\infty} 2a_{n-2} x^{n-1} \\
&= 0a_0 x^{-1} + a_1 x^0 + \sum_{n=2}^{\infty} a_n n x^{n-1} - \sum_{n=2}^{\infty} 2a_{n-2} x^{n-1} \\
&= 0a_0 x^{-1} + a_1 x^0 + \sum_{n=2}^{\infty} \big[a_n n - 2a_{n-2}\big] x^{n-1}.
\end{aligned}$$

The first term is zero for all nonzero values of x so we see that a_0 can be any real number, i.e. it is an arbitrary constant. But the term a_1 must be zero since the constant on the left hand side of the equation is zero, i.e., $a_1 = 0$.

Now, in order that the remaining equation

$$\sum_{n=2}^{\infty} \big[a_n n - 2a_{n-2}\big] x^{n-1} = 0$$

holds for all $x \neq 0$ we would need

$$\big[a_n n - 2a_{n-2}\big] = 0, \quad n = 2, 3, \cdots .$$

This is the same as

$$a_n = \frac{2a_{n-2}}{n}, \quad n = 2, 3, \cdots \quad \text{(Recursion Formula).}$$

The Recursion Formula can be used to successively obtain all the even terms a_n (for n even) in terms of a_0 and we can see that all the odd terms a_n (for n odd) must be 0 since $a_1 = 0$. Thus we know that

$$a_{2k+1} = 0, \quad k = 1, 2, \cdots .$$

Note that if $y = \sum_{n=0}^{\infty} a_n x^n$ then $y(0) = a_0$ so we have $a_0 = 1$ and

$$\boldsymbol{n = 2}, \qquad a_2 = \frac{2a_0}{2}$$

$$\boldsymbol{n = 4}, \qquad a_4 = \frac{2a_2}{4} = \frac{2^2 a_0}{4 \cdot 2} = \frac{2^2}{4 \cdot 2}$$

$$\boldsymbol{n = 6}, \qquad a_6 = \frac{2a_4}{6} = \frac{2^3}{6 \cdot 4 \cdot 2}$$

$$\boldsymbol{n = 2k}, \qquad a_{2k} = \frac{2a_{2k-2}}{(2k)} = \frac{2^k}{(2k) \cdot (2k-2) \cdots 4 \cdot 2}$$

$$= \frac{2^k}{2^k\, k!} = \frac{1}{k!}$$

It is easy to see the pattern and we can extrapolate the above to conclude that

$$a_{2k} = \frac{1}{k!} \quad \text{for all} \quad k = 1, 2, \cdots .$$

We conclude that

$$y = a_0 + \sum_{k=1}^{\infty} a_{2k} x^{2k} = a_0 + a_0 \sum_{k=1}^{\infty} \frac{1}{k!} x^{2k} = \sum_{k=0}^{\infty} \frac{(x^2)^k}{k!} = e^{x^2}.$$

Where we have used the first entry in our table of power series examples with x^2 instead of x.

Notice this is a first order linear initial value problem. We could find a general solution using the integration factor e^{-x^2}. to get

$$\left[e^{-x^2}y\right]' = 0 \quad \Rightarrow \quad e^{-x^2}y = C \quad \Rightarrow \quad y = Ce^{x^2}.$$

Example 5.4. Find the general solution of $(x^2+1)y'' - 4xy' + 6y = 0$ in the form $y = \sum_{n=0}^{\infty} a_n x^n$. Notice that near $x = 0$ the leading coefficient is not zero and all coefficients are analytic so we can expect to have a power series solution.

First we compute $y' = \sum_{n=0}^{\infty} a_n n x^{n-1}$ and $y'' = \sum_{n=0}^{\infty} a_n n(n-1)x^{n-2}$. Then we substitute these series into the differential equation and try to find the coefficients a_n so that the resulting equation is satisfied. We have

$$\begin{aligned}
0 &= (x^2+1)y'' - 4xy' + 6y = (x^2+1)\sum_{n=0}^{\infty} a_n n(n-1)x^{n-2} - 4x\sum_{n=0}^{\infty} a_n n x^{n-1} + 6\sum_{n=0}^{\infty} a_n x^n \\
&= \sum_{n=0}^{\infty} a_n n(n-1)x^{n-2} + \sum_{n=0}^{\infty}\big[n(n-1) - 4n + 6\big]a_n x^n \\
&= 0a_0x^{-2} + 0a_1x^{-1} + \sum_{n=2}^{\infty} a_n n(n-1)x^{n-2} + \sum_{n=0}^{\infty}(n^2 - 5n + 6)a_n x^n \\
&= 0a_0x^{-2} + 0a_1x^{-1} + \sum_{n=2}^{\infty} a_n n(n-1)x^{n-2} + \sum_{n=2}^{\infty}(n^2 - 9n + 20)a_{n-2}x^{n-2} \\
&= 0a_0x^{-2} + 0a_1x^{-1} + \sum_{n=2}^{\infty}\big[a_n n(n-1) + (n-4)(n-5)a_{n-2}\big]x^{n-2}.
\end{aligned}$$

The first two terms are zero for all nonzero values of x so we see that a_0 and a_1 can be any real numbers, i.e. they are arbitrary constants.

Now, in order that the remaining equation

$$\sum_{n=2}^{\infty}\big[a_n n(n-1) + (n-4)(n-5)a_{n-2}\big]x^{n-2} = 0$$

holds for all $x \neq 0$ we would need

$$a_n n(n-1) + (n-4)(n-5)a_{n-2} = 0, \quad n = 2, 3, \cdots.$$

This is the same as

$$a_n = \frac{-(n-4)(n-5)a_{n-2}}{n(n-1)}, \quad n = 2, 3, \cdots \quad \text{(Recursion Formula)}.$$

The Recursion Formula can be used to successively obtain the terms a_n in terms of a_0 and a_1 as follows. In this present case we see that the

$$\boldsymbol{n=2}, \quad a_2 = \frac{-(-2)(-3)a_0}{2\cdot 1} \qquad\qquad \boldsymbol{n=3}, \quad a_3 = \frac{-(-1)(-2)a_1}{3\cdot 2}$$

$$= -3a_0 \qquad\qquad\qquad\qquad = \frac{-a_1}{3}$$

$$\boldsymbol{n=4}, \quad a_4 = \frac{-(0)(-1)a_2}{4\cdot 3} \qquad\qquad \boldsymbol{n=5}, \quad a_5 = \frac{-(1)(0)a_3}{5\cdot 4}$$

$$= 0 \qquad\qquad\qquad\qquad = 0$$

$$\boldsymbol{n=6}, \quad a_6 = 0 \qquad\qquad \boldsymbol{n=7}, \quad a_7 = 0$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

It is easy to see that for $n = 2k$ with $k \geq 2$ we have $a_{2k} = 0$ and for $n = 2k+1$ with $k \geq 2$ we have $a_{2k+1} = 0$. So we have

$$y = \sum_{n=0}^{\infty} a_n x^n = a_0(1 - 3x^2) + a_1 \left(x - \frac{1}{3}x^3 \right).$$

In the next example we consider a problem which looks very much like the previous one but the answer ends up quite different.

Example 5.5. Find the general solution of $(1-x^2)y'' - 6xy' - 4y = 0$ in the form $y = \sum_{n=0}^{\infty} a_n x^n$. Notice that near $x = 0$ the leading coefficient is not zero and all coefficients are analytic so we can expect to have a power series solution.

First we compute $y' = \sum_{n=0}^{\infty} a_n n x^{n-1}$ and $y'' = \sum_{n=0}^{\infty} a_n n(n-1)x^{n-2}$. Then we substitute these series into the differential equation and try to find the coefficients a_n so that the resulting equation is satisfied. We have

$$
\begin{aligned}
0 &= (1-x^2)y'' - 6xy' - 4y \\
&= (1-x^2)\sum_{n=0}^{\infty} a_n n(n-1)x^{n-2} - 6x\sum_{n=0}^{\infty} a_n n x^{n-1} - 4\sum_{n=0}^{\infty} a_n x^n \\
&= \sum_{n=0}^{\infty} a_n n(n-1)x^{n-2} - \sum_{n=0}^{\infty} \big[n(n-1) + 6n + 4\big] a_n x^n \\
&= 0a_0x^{-2} + 0a_1x^{-1} + \sum_{n=2}^{\infty} a_n n(n-1)x^{n-2} - \sum_{n=0}^{\infty}(n^2+5n+4)a_n x^n \\
&= 0a_0x^{-2} + 0a_1x^{-1} + \sum_{n=2}^{\infty} a_n n(n-1)x^{n-2} - \sum_{n=2}^{\infty}(n^2+n-2)a_{n-2} x^{n-2} \\
&= 0a_0x^{-2} + 0a_1x^{-1} + \sum_{n=2}^{\infty} a_n n(n-1)x^{n-2} - \sum_{n=2}^{\infty}(n-1)(n+2)a_{n-2} x^{n-2} \\
&= 0a_0x^{-2} + 0a_1x^{-1} + \sum_{n=2}^{\infty}(n-1)\big[a_n n + (n+2)a_{n-2}\big] x^{n-2}.
\end{aligned}
$$

The first two terms are zero for all nonzero values of x so we see that a_0 and a_1 can be any real numbers, i.e. they are arbitrary constants.

Now, in order that the remaining equation

$$\sum_{n=2}^{\infty}(n-1)\big[a_n n - (n+2)a_{n-2}\big] x^{n-2} = 0$$

holds for all $x \neq 0$ we would need

$$a_n n - (n+2)a_{n-2} = 0, \quad n = 2, 3, \cdots .$$

This is the same as

$$a_n = \frac{(n+2)a_{n-2}}{n}, \quad n = 2, 3, \cdots \text{ (Recursion Formula).}$$

The Recursion Formula can be used to successively obtain the terms a_n in terms of a_0

and a_1 as follows. In this present case we see that the

$$\boldsymbol{n=2}, \qquad a_2 = \frac{4\ a_0}{2} \qquad\qquad \boldsymbol{n=3}, \qquad a_3 = \frac{5\ a_1}{3}$$

$$\boldsymbol{n=4}, \qquad a_4 = \frac{6a_2}{4} = \frac{6\cdot 4\ a_0}{4\cdot 2} \qquad\qquad \boldsymbol{n=5}, \qquad a_5 = \frac{7\ a_1}{5} = \frac{7\cdot 5\ a_1}{5\cdot 3}$$

$$\boldsymbol{n=6}, \qquad a_6 = \frac{8\ a_4}{6} = \frac{8\cdot 6\cdot 4\ a_0}{6\cdot 4\cdot 2} \qquad\qquad \boldsymbol{n=7}, \qquad a_7 = \frac{9\ a_5}{7} = \frac{9\cdot 7\cdot 5\ a_1}{7\cdot 5\cdot 3}$$

It is easy to see that for $n = 2k$ with $k \geq 2$ we have

$$a_{2k} = \frac{(2k+2)(2k)\cdots(6)(4)}{(2k)(2k-2)\cdots(4)(2)}\ a_0 = \frac{2^k(k+1)!}{2^k k!}\ a_0 = (k+1)\ a_0$$

and for $n = 2k+1$ with $k \geq 2$ we have

$$a_{2k+1} = \frac{(2k+3)(2k+1)\cdots(7)(5)}{(2k+1)(2k-1)\cdots(5)(3)}\ a_1 = \frac{(2k+3)}{3}\ a_1.$$

So we have

$$y = \sum_{n=0}^{\infty} a_n x^n = \left[a_0 + \sum_{k=1}^{\infty}(k+1)\ x^{2k}\right] + \left[a_1 x + \frac{1}{3}\sum_{k=1}^{\infty}(2k+3)\ x^{2k+1}\right].$$

With a little work it can be shown (using geometric series arguments) that for $|x| < 1$ the above sums reduce to

$$y = a_0\,\frac{1}{(1-x^2)^2} + a_1\,\frac{(3x - x^3)}{3(1-x^2)^2}.$$

Namely, let us define $v = x^2$ so that

$$y_1 = a_0 + a_0 \sum_{k=1}^{\infty}(k+1)x^{2k} = \sum_{k=0}^{\infty}(k+1)v^k$$

$$
\begin{aligned}
&= a_0 \frac{d}{dv}\left(\sum_{k=0}^{\infty} v^{k+1}\right) = \frac{d}{dv}\left(\frac{v}{1-v}\right) \\
&= a_0 \frac{1\cdot(1-v) - v\cdot(-1)}{(1-v)^2} = \frac{1}{(1-v)^2} \\
&= \frac{a_0}{(1-x^2)^2}.
\end{aligned}
$$

For y_2 we have

$$
\begin{aligned}
y_2 &= a_1 x + a_1 \frac{1}{3}\sum_{k=1}^{\infty}(2k+3)x^{2k+1} \\
&= a_1 \frac{1}{3}\sum_{k=0}^{\infty}(2k+3)x^{2k+1} = a_1 \frac{1}{3x}\sum_{k=0}^{\infty}(2k+3)x^{2k+2} \\
&= a_1 \frac{1}{3x}\frac{d}{dx}\left(\sum_{k=0}^{\infty} x^{2k+3}\right) = a_1 \frac{1}{3x}\frac{d}{dx}\left(x^3\sum_{k=0}^{\infty}(x^2)^k\right) \\
&= a_1 \frac{1}{3x}\frac{d}{dx}\left(\frac{x^3}{1-x^2}\right) = a_1 \frac{1}{3x}\left(\frac{3x^2 - x^4}{(1-x^2)^2}\right) \\
&= a_1 \frac{3x - x^3}{3(1-x^2)^2}.
\end{aligned}
$$

Next we consider a second order IVP.

Example 5.6. Find the solution of the IVP $y'' - 2y' + y = 0$ with $y(0) = 0$ and $y'(0) = 1$ in the form $y = \sum_{n=0}^{\infty} a_n x^n$. Notice that near $x = 0$ the leading coefficient is not zero and all coefficients are analytic so we can expect to have a power series solution.

First we compute $y' = \sum_{n=0}^{\infty} a_n n x^{n-1}$ and $y'' = \sum_{n=0}^{\infty} a_n n(n-1)x^{n-2}$. Then we substitute these series into the differential equation and try to find the coefficients a_n so that the resulting equation is satisfied. We have

$$
\begin{aligned}
0 &= y'' - 2y' + y \\
&= \sum_{n=0}^{\infty} a_n n(n-1)x^{n-2} - 2\sum_{n=0}^{\infty} a_n n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n \\
&= \sum_{n=0}^{\infty} a_n n(n-1)x^{n-2} - 2\sum_{n=0}^{\infty} a_n n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n \\
&= 0a_0 x^{-2} + 0a_1 x^{-1} + \sum_{n=2}^{\infty} a_n n(n-1)x^{n-2} + \sum_{n=1}^{\infty} a_{n-1}(n-1)x^{n-2} + \sum_{n=2}^{\infty} a_{n-2}x^{n-2}
\end{aligned}
$$

$$= 0a_0x^{-2} + 0a_1x^{-1} + + \sum_{n=2}^{\infty} a_n n(n-1)x^{n-2} - 2a_0(0)x^{-1} + -2\sum_{n=2}^{\infty} a_{n-1}(n-1)x^{n-2} + \sum_{n=2}^{\infty} a_{n-2}x^{n-2}$$

$$= 0a_0x^{-2} + [0a_1 - 2a_0(0)]x^{-1} + \sum_{n=2}^{\infty}[a_n n(n-1) - 2a_{n-1}(n-1) + a_{n-2}]x^{n-2}$$

The first two terms are zero for all nonzero values of x so we see that a_0 and a_1 can be any real numbers, i.e. they are arbitrary constants. But according to the given initial conditions we can say that $a_0 = 0$ and $a_1 = 1$.

Now, in order that the remaining equation

$$\sum_{n=2}^{\infty}[a_n n(n-1) - 2a_{n-1}(n-1) + a_{n-2}]x^{n-2} = 0$$

holds for all $x \neq 0$ we would need

$$a_n n(n-1) - 2a_{n-1}(n-1) + a_{n-2} = 0, \quad n = 2, 3, \cdots .$$

This is the same as

$$a_n = \frac{2(n-1)a_{n-1} - a_{n-2}}{n(n-1)}, \quad n = 2, 3, \cdots \quad \text{(Recursion Formula)}.$$

The Recursion Formula can be used to successively obtain the terms a_n in terms of a_0 and a_1 as follows. In this present case we see that the coefficients do no break up into the even and odd terms.

$$\boldsymbol{n=2}, \quad a_2 = \frac{2(1)a_1 - a_0}{2 \cdot 1} = 1$$
$$\boldsymbol{n=3}, \quad a_3 = \frac{2(2)a_2 - a_1}{3 \cdot 2} = \frac{4-1}{3 \cdot 2} = \frac{1}{2!}$$
$$\boldsymbol{n=4}, \quad a_4 = \frac{2(3)(1/2!) - 1}{4 \cdot 3} = \frac{1}{3!}$$
$$\boldsymbol{n=5}, \quad a_4 = \frac{2(4)a_4 - a_3}{5 \cdot 4} = \frac{5}{5!} = \frac{1}{4!}$$

Writing out a few more if necessary you can conclude that

$$a_n = \frac{1}{(n-1)!} \quad n \geq 2.$$

Thus we obtain

$$\begin{aligned} y &= a_0 + a_1 x + \sum_{n=2}^{\infty} a_n x^n = x + \sum_{n=2}^{\infty} \frac{x^n}{(n-1)!} \\ &= x + \sum_{n=1} \frac{x^{n+1}}{n!} = x\left(1 + \sum_{n=1} \frac{x^n}{n!}\right) \\ &= x \sum_{n=0} \frac{x^n}{n!} = xe^x. \end{aligned}$$

Notice back in Chapter 3 when we worked a problem like this it went as follows. Consider the auxiliary equation

$$0 = r^2 - 2r + 1 = (r-1)^2$$

which implies $r = 1, 1$, a double root. Therefore $y = c_1 e^x + c_2 x e^x$ and we need $y' = c_1 e^x + c_2(x+1)e^x$. Apply the initial conditions to get $c_1 = 0$ and $c_2 = 1$ so we have $y = xe^x$.

Example 5.7. Find the general solution of $y'' - xy' - y = 0$ in the form $y = \sum_{n=0}^{\infty} a_n x^n$. Notice that near $x = 0$ the leading coefficient is not zero and all coefficients are analytic so we can expect to have a power series solution.

First we compute $y' = \sum_{n=0}^{\infty} a_n n x^{n-1}$ and $y'' = \sum_{n=0}^{\infty} a_n n(n-1)x^{n-2}$. Then we substitute these series into the differential equation and try to find the coefficients a_n so that the resulting equation is satisfied. We have

$$\begin{aligned} 0 &= y'' - xy' - y \\ &= \sum_{n=0}^{\infty} a_n n(n-1)x^{n-2} - x\sum_{n=0}^{\infty} a_n n x^{n-1} - \sum_{n=0}^{\infty} a_n x^n \\ &= \sum_{n=0}^{\infty} a_n n(n-1)x^{n-2} - \sum_{n=0}^{\infty} a_n n x^n + \sum_{n=0}^{\infty} a_n x^n \end{aligned}$$

$$
\begin{aligned}
&= \sum_{n=0}^{\infty} a_n n(n-1)x^{n-2} - \sum_{n=0}^{\infty}(n+1)a_n x^n \\
&= 0a_0x^{-2} + 0a_1x^{-1} + \sum_{n=0}^{\infty} a_{n-1}(n-1)x^{n-2} - \sum_{n=2}^{\infty}(n-1)a_{n-2}x^{n-2} \\
&= 0a_0x^{-2} + 0a_1x^{-1} + \sum_{n=2}^{\infty}[a_n n(n-1) - (n-1)a_{n-2}]x^{n-2}
\end{aligned}
$$

The first two terms are zero for all nonzero values of x so we see that a_0 and a_1 can be any real numbers, i.e. they are arbitrary constants. But according to the given initial conditions we can say that $a_0 = 0$ and $a_1 = 1$.

Now, in order that the remaining equation

$$
\sum_{n=2}^{\infty}[a_n n(n-1) - (n-1)a_{n-2}]x^{n-2} = 0
$$

holds for all $x \neq 0$ we would need

$$
\big[a_n n(n-1) - (n-1)a_{n-2}\big] = 0, \quad n = 2, 3, \cdots .
$$

This is the same as

$$
a_n = \frac{a_{n-2}}{n}, \quad n = 2, 3, \cdots \text{ (Recursion Formula).}
$$

The Recursion Formula can be used to successively obtain the terms a_n in terms of a_0 and a_1 as follows. In this present case we see that the

Once again as in some earlier examples the coefficients break up into the even and odd terms.

$$
\begin{aligned}
&\boldsymbol{n = 2}, \quad a_2 = \frac{a_0}{2} \\
&\boldsymbol{n = 4}, \quad a_4 = \frac{a_2}{4} = \frac{a_0}{4 \cdot 2} \\
&\boldsymbol{n = 6}, \quad a_6 = \frac{a_4}{6} = \frac{a_0}{6 \cdot 4 \cdot 2}
\end{aligned}
$$

$$\boldsymbol{n=2k}, \quad a_{2k} = \frac{a_{2k-2}}{2k} = \frac{a_0}{2^k k!}$$

Writing out a few more if necessary you can conclude that

$$a_{2k} = \frac{a_0}{2^k k!} \quad k \geq 1.$$

$$\begin{aligned}
\boldsymbol{n=3}, &\quad a_3 = \frac{a_1}{3} \\
\boldsymbol{n=5}, &\quad a_5 = \frac{a_3}{5} = \frac{a_1}{5 \cdot 3} \\
\boldsymbol{n=7}, &\quad a_7 = \frac{a_5}{7} = \frac{a_1}{7 \cdot 5 \cdot 1} \\
\boldsymbol{n=2k+1}, &\quad a_{2k+1} = \frac{a_{2k-1}}{2k+1} = \frac{a_1}{(2k+1)(2k-1)\cdots 5 \cdot 3} \\
&\quad = \frac{2^k k! a_1}{(2k+1)!}
\end{aligned}$$

The solution can be written in the form

$$y = a_0 y_1 + a_1 y_2$$

where

$$y_1 = a_0 + \sum_{k=1}^{\infty} \frac{a_0 x^{2k}}{2^k k!} = a_0 \sum_{k=0}^{\infty} \frac{\left(\frac{x^2}{2}\right)^k}{k!} = e^{x^2/2}.$$

The odd terms present a bigger challenge and can only give a function in closed form in terms of the error function

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt.$$

We get

$$y_2 = \text{erf}\left(\frac{x}{\sqrt{2}}\right) e^{x^2/2}.$$

Let us consider an example that could easily be on an exam.

Example 5.8. Find the solution of the IVP $(2-x)y'+3y=0$ with $y(0)=8$ in the form $y=\sum_{n=0}^{\infty}a_nx^n$. All the coefficients are analytic but the leading coefficient can be 0 when $x=2$ so we need to make sure we say away from $x=2$ but otherwise we can expect to have a convergent power series solution in $-2<x<2$.

First we compute $y'=\sum_{n=0}^{\infty}a_nnx^{n-1}$. Then we substitute these series into the differential equation and try to find the coefficients a_n so that the resulting equation is satisfied. We have

$$\begin{aligned}
0 &= (2-x)\sum_{n=0}^{\infty}a_nnx^{n-1}+3\sum_{n=0}^{\infty}a_nx^n \\
&= 2\sum_{n=0}^{\infty}a_nnx^{n-1}-\sum_{n=0}^{\infty}a_nnx^n+3\sum_{n=0}^{\infty}a_nx^n \\
&= \sum_{n=0}^{\infty}2a_nnx^{n-1}-\sum_{n=0}^{\infty}a_n(n-3)x^n \\
&= \sum_{n=0}^{\infty}2a_nnx^{n-1}-\sum_{n=1}^{\infty}a_{n-1}(n-4)x^{n-1} \\
&= 2a_0\cdot 0x^{-1}+\sum_{n=1}^{\infty}[2a_nn-(n-4)a_{n-1}]x^{n-1}
\end{aligned}$$

From this we conclude a_0 is arbitrary and $[2a_nn-(n-4)a_{n-1}]=0$ for all $n\ge 1$. Thus we obtain the recursion formula

$$a_n=\frac{(n-4)a_{n-1}}{2n} \quad n=1,2,\cdots.$$

Together with the initial condition $a_0=y(0)=8$ this gives us

$$\begin{aligned}
&\boldsymbol{n=1},\ a_1=\frac{a_0(1-4)}{2\cdot 1}=-12 \\
&\boldsymbol{n=2},\ a_2=\frac{a_1(2-4)}{2\cdot 2}=\frac{-12(-2)}{2\cdot 2}=6 \\
&\boldsymbol{n=3},\ a_3=\frac{a_2(3-4)}{2\cdot 3}=\frac{-6}{6}=-1 \\
&\boldsymbol{n=4},\ a_4=\frac{a_3(4-4)}{2\cdot 4}=0
\end{aligned}$$

Therefore all $a_n = 0$ for $n \geq 4$. Our series solution becomes a polynomial

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 = 8 - 12x + 6x^2 - x^3 = (2-x)^3.$$

Here is a another very similar looking example.

Example 5.9. Find the solution of the IVP $(2+x)y' + y = 0$ with $y(0) = 1$ in the form $y = \sum_{n=0}^{\infty} a_n x^n$. All the coefficients are analytic but the leading coefficient can be 0 when $x = -2$ so we need to make sure we say away from $x = 2$ but otherwise we can expect to have a convergent power series solution in $-2 < x < 2$.

First we compute $y' = \sum_{n=0}^{\infty} a_n n x^{n-1}$. Then we substitute these series into the differential equation and try to find the coefficients a_n so that the resulting equation is satisfied. We have

$$\begin{aligned}
0 &= (2+x)\sum_{n=0}^{\infty} a_n n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n \\
&= 2\sum_{n=0}^{\infty} a_n n x^{n-1} + \sum_{n=0}^{\infty} a_n n x^n + \sum_{n=0}^{\infty} a_n x^n \\
&= \sum_{n=0}^{\infty} 2a_n n x^{n-1} + \sum_{n=0}^{\infty} a_n (n+1) x^n \\
&= \sum_{n=0}^{\infty} 2a_n n x^{n-1} + \sum_{n=1}^{\infty} a_{n-1} (n-4) x^{n-1} \\
&= 2a_0 \cdot 0 x^{-1} + \sum_{n=1}^{\infty} [2a_n n - n a_{n-1}] x^{n-1}
\end{aligned}$$

From this we conclude a_0 is arbitrary and $[2a_n n + n a_{n-1}] = 0$ for all $n \geq 1$. Thus we obtain the recursion formula

$$a_n = \frac{-a_{n-1}}{2} \quad n = 1, 2, \cdots .$$

Together with the initial condition $a_0 = y(0) = 1$ this gives us

$$\boldsymbol{n=1}, \qquad a_1 = \frac{-a_0}{2} = -\frac{1}{2}$$

$$\boldsymbol{n=2}, \quad a_2 = \frac{-a_1}{2} = \left(\frac{-1}{2}\right)^2$$

$$\boldsymbol{n=3}, \quad a_3 = \frac{-a_2}{2} = \left(\frac{-1}{2}\right)^3$$

$$\boldsymbol{n=4}, \quad a_4 = \frac{-a_3)}{2} = \left(\frac{-1}{2}\right)^4$$

So in general we get

$$a_n = \left(\frac{-1}{2}\right)^n \quad n = 1, 2, \cdots .$$

Therefore all $a_n = 0$ for $n \geq 4$. Our series solution becomes a polynomial

$$y = \sum_{n=0}^{\infty} a_n x^n = 1 + \sum_{n=1}^{\infty} \left(\frac{-1}{2}\right)^n x^n = \sum_{n=0}^{\infty} \left(\frac{-x}{2}\right)^n .$$

This is a geometric series with common ration $\left(\frac{-x}{2}\right)$ which is convergent so long as $\left|\frac{-x}{2}\right| < 1$ or $|x| < 2$. And we get

$$y = \frac{1}{1 - \left(\frac{-x}{2}\right)} = \frac{2}{2+x}.$$

We consider one final example from mathematical physics - the Airy Equation. Like the last example, this is a problem that we could not have solved using any previous methods covered in the class. As simple as it looks this is example produces the most complicated answer we have considered thus far.

Example 5.10. Find the general solution of $y'' - xy = 0$ in the form $y = \sum_{n=0}^{\infty} a_n x^n$. All the coefficients are analytic and we can expect to have a power series solution.

First we compute $y' = \sum_{n=0}^{\infty} a_n n x^{n-1}$ and $y'' = \sum_{n=0}^{\infty} a_n n(n-1) x^{n-2}$. Then we substitute these series into the differential equation and try to find the coefficients a_n so that the resulting equation is satisfied. We have

$$
\begin{aligned}
0 = y'' - xy &= \sum_{n=0}^{\infty} a_n n(n-1)x^{n-2} - x\sum_{n=0}^{\infty} a_n x^n \\
&= \sum_{n=0}^{\infty} a_n n(n-1)x^{n-2} - \sum_{n=0}^{\infty} a_n x^{n+1} \\
&= 0a_0x^{-2} + 0a_1x^{-1} + 2a_2x^0 + \sum_{n=3}^{\infty} a_n n(n-1)x^{n-2} - \sum_{n=3}^{\infty} a_{n-3}x^{n-2} \\
&= 0a_0x^{-2} + 0a_1x^{-1} + 2a_2x^0 + \sum_{n=2}^{\infty} \big[a_n n(n-1) - a_{n-3}\big]x^{n-2}.
\end{aligned}
$$

The first two terms are zero for all nonzero values of x so we see that a_0 and a_1 can be any real numbers, i.e. they are arbitrary constants. But, in order that the third term be zero we must have $a_2 = 0$.

Then, in order that the remaining equation

$$\sum_{n=2}^{\infty} \big[a_n n(n-1) - a_{n-3}\big]x^{n-2} = 0$$

holds for all $x \neq 0$ we would need

$$\big[a_n n(n-1) - a_{n-3}\big] = 0, \quad n = 2, 3, \cdots .$$

This is the same as

$$a_n = \frac{a_{n-3}}{n(n-1)}, \quad n = 3, 4, \cdots \quad \text{(Recursion Formula)}.$$

The Recursion Formula can be used to successively obtain the terms a_n in terms of a_0 and a_1 as follows. In this present case we see that the a_0 and a_1 are arbitrary constants and $a_2 = 0$.

For this example we see that the coefficients break up into three groups:

1. terms that begin with a_0 and differ by an index of 3, i.e., a_{3k} for $k = 1, 2, \cdots$.

2. terms that begin with a_1 and differ by an index of 3, i.e., a_{3k+1} for $k = 1, 2, \cdots$.

3. terms that begin with a_2 and differ by an index of 3, i.e., a_{3k+2} for $k = 1, 2, \cdots$.

Notice that since $a_2 = 0$ we see that all the terms in the group $a_{3k+2} = 0$. So we only need to consider terms of the form a_{3k} and terms of the form a_{3k+1} for $k = 0, 1, \cdots$.

With some work we can compute the general terms in each case:

1. For $k = 1, 2, \cdot$

$$a_{3k} = \frac{a_0}{(2 \cdot 3)(5 \cdot 6) \cdots (3k-1)(3k)}.$$

2. For $k = 1, 2, \cdot$

$$a_{3k+1} = \frac{a_1}{(3 \cdot 4)(6 \cdot 7) \cdots (3k)(3k+1)}.$$

So we have

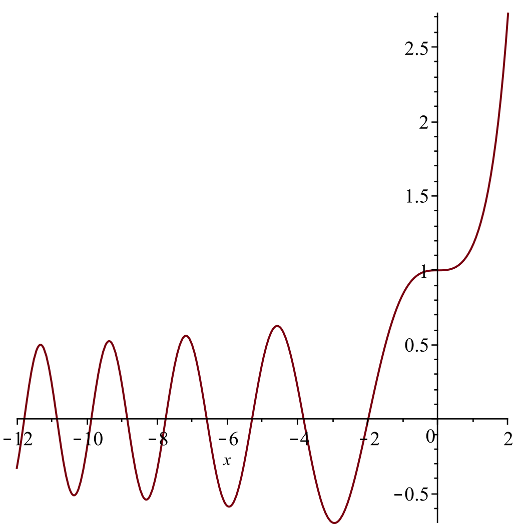
$$y = a_0 y_1 + a_1 y_2$$

where

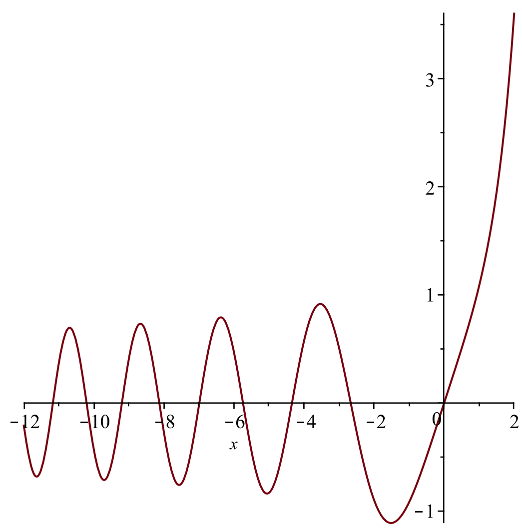
$$y_1 = \left[1 + \sum_{k=1}^{\infty} \frac{x^{3k}}{(2 \cdot 3)(5 \cdot 6) \cdots (3k-1)(3k)}\right]$$

and

$$y_2 = \left[x + \sum_{k=1}^{\infty} \frac{x^{3k+1}}{(3 \cdot 4)(6 \cdot 7) \cdots (3k)(3k+1)}\right].$$



y_1



y_2