Chapter 3

3 Introduction

Notes Proofread by Yunting Gao and corrections made on 03/12/2021

3.1 Theory of Linear Equations

Recall that an nth order *Linear* ODE is an equation that can be written in the form

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{d^1 y}{dx^1} + a_0(x)y(x) = g(x). \tag{1}$$

Recall that we say (1) is *homogeneous* if $g(x) = 0$ and *nonhomogeneous* if $g(x) \neq 0$

The Initial Value Problem (IVP) is given by (1) together with a set of n *initial conditions*

$$y(x_0) = y_1, \quad y'(x_0) = y_2, \quad \cdots, \quad y^{(n-1)}(x_0) = y_n. \tag{ICS}$$

Theorem 3.1 (Existence Uniqueness for Linear IVPs)**.** *If the functions $\{a_j(x)\}_{j=0}^n$ and $g(x)$ are continuous on an interval $I = \{x : a < x < b\}$ and $a_n(x) \neq 0$ for all $x \in I$ and $x_0 \in I$, then there is a unique solution $y = \varphi(x)$ for all $x \in I$.*

Example 3.1. Consider the initial value problem $(x-2)y'' + 3y = x$ with ICS $y(0) = 0$ and $y'(0) = 1$. Here the leading coefficient is $a_2 = (x-2)$ which satisfies $a_2(x) = 0$ when $x = 2$. Now the initial $x_0 = 0$ lies to the left of $x = 2$. So by Theorem 3.1 we see that a unique solution exists on the interval $-\infty < x < 2$.

Remark 3.1. It is sometimes useful to use the following notation. Let $D = d/dx$ denote the derivative thought of as an operator. This notation allows us to define an operator

$$L = (a_n(x)D^n + a_{n-1}(x)D^{n-1} + \cdots + a_1(x)D + a_0(x))$$

which we can use to write (1) as

$$Ly(x) = g(x).$$

With this notation we can write, in a very simple form, the important defining property of a Linear equation. If $f(x)$ and $g(x)$ are two functions and α and β are two constants then we have

$$L(\alpha f(x) + \beta g(x)) = \alpha L(f(x)) + \beta L(g(x)).$$

As a result of the linearity expressed above we can state the *Principle of Superposition* for linear equations as follows: If $y_1, y_2, \cdots, y_n$ are n functions satisfying the homogeneous problem $Ly = 0$ then $y = c_1y_1 + c_2y_2 + \cdots + c_ny_n$ is also a solution.

Definition 3.1. A set of functions $y_1, y_2, \cdots, y_n$ are called *Linearly Independent* on an interval $I = (a, b)$ if

$$c_1y_1(x) + c_2y_2(x) + \cdots + c_ny_n(x) = 0 \ \ \forall \ \ x \in I \quad \Leftrightarrow \quad c_1 = c_2 = \cdots = c_n = 0.$$

If the functions are not linearly independent then we say they are *Dependent*. This means that there must exist a set of constants $c_1, c_2, \cdots, c_n$ not all zero so that

$$c_1y_1(x) + c_2y_2(x) + \cdots + c_ny_n(x) = 0 \ \ \forall \ \ x \in I.$$

Example 3.2.

1. The functions $1, x, x^2, \cdots, x^n$ are linearly independent since a linear combination

$$c_1 + c_2x + \cdots + c_nx^n$$

is a polynomial of degree n which can have at most n real roots so it cannot be identically 0 in any interval unless $c_1 = c_2 = \cdots = c_n = 0$.

2. The functions $x, |x|$ are linearly independent on $\mathbb{R} = (-\infty, \infty)$ but not on the interval $(0, \infty)$.

3. To show the functions $y_1 = \sin(x)$ and $y_2 = \cos(x)$ are linearly independent we consider $c_1 \sin(x) + c_2 \cos(x) = 0$. If we suppose (by way of contradiction) that $c_1 \neq 0$ then we can divide by c_1 and divide by $\cos(x)$ to write

$$\tan(x) = -\frac{c_2}{c_1}$$

but notice that the left side is the well known function $\tan(x)$ which is not constant, while the right hand side is a constant. This is a contradiction, which implies that our assumption that $c_1 \neq 0$ is false so we must have $c_1 = 0$. But then we are left with $c_2 \cos(x) = 0$ for all x which again is only possible if $c_2 = 0$. We conclude that $c_1 = c_2 = 0$ and trhe functions are linearly independent.

The above examples suggest that deciding whether functions are dependent or independent can be difficult. We now present a simple method for deciding linear dependence or independence.

Definition 3.2. Given a set of functions $y_1, y_2, \cdots, y_n$ we define the *Wronskian* by

$$W = W(y_1, y_2, \cdots, y_n) = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \vdots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix} \tag{2}$$

Here the notation $|\cdot|$ denotes the determinant of the $n \times n$ matrix.

Theorem 3.2. *[Wronskian Test for Independence] If $y_1, y_2, \cdots, y_n$ are n solutions to an nth order linear homogeneous equation on an interval I. Then the functions are linearly independent $\Leftrightarrow$ $W(y_1, y_2, \cdots, y_n)(x) \neq 0$ for every $x \in I$.*

Remark 3.2. More generally, if the Wronskian of any set of n functions is not zero on an interval I then the functions are linearly independent on the interval I.

In the case $n = 2$ the Wronskian of two functions y_1, y_2 is

$$W = W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'.$$

In the case $n = 3$ the Wronskian of three functions y_1, y_2, y_3 is

$$W = W(y_1, y_2, y_3) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}.$$

More generally, determinants are defined in Chapter 8 Section 4 where they describe the concepts of minors and cofactors. As an example we give the expansion by expansion by minors and cofactors using the first row. Consider the determinant of a 3×3 matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

First cover the row with plus and minus signs beginning with a $+$ in the $(1,1)$ position and then alternating signs. Then take the sum of the products of the sign, the element of the row and the determinant of the 2×2 matrix obtained by deleting the row and column that intersect in that particular element.

$$\begin{aligned} \det(A) &= \begin{vmatrix} a_{11}^{+} & a_{12}^{-} & a_{13}^{+} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \\ &= a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}. \end{aligned}$$

If the functions are solutions of a linear homogeneous ODE then the functions are linearly independent on an interval I if and only if the Wronskian is not zero at a single $x \in I$ (and therefore for all $x \in I$).

Example 3.3. Show that $y_1 = e^{-3x}$ and $y_2 = e^{4x}$ are linearly independent for $x > 0$

$$W = W(y_1, y_2) = \begin{vmatrix} e^{-3x} & e^{4x} \\ -3e^{-3x} & 4e^{4x} \end{vmatrix} = 4e^{x} + 3e^{x} = 7e^{x} \neq 0.$$

Example 3.4. Let us reconsider showing $y_1 = \sin(x)$ and $y_2 = \cos(x)$ are linearly independent

$$W = W(y_1, y_2) = \begin{vmatrix} \sin(x) & \cos(x) \\ \cos(x) & -\sin(x) \end{vmatrix} = -\sin^2(x) - \cos^2(x) = -1 \neq 0.$$

Example 3.5. Show that $y_1 = e^{-3x}$ and $y_2 = e^{4x}$ are linearly independent for all x

$$W = W(y_1, y_2) = \begin{vmatrix} e^{-3x} & e^{4x} \\ -3e^{-3x} & 4e^{4x} \end{vmatrix} = 4e^x + 3e^x = 7e^x \neq 0.$$

Example 3.6.

1. Consider the three functions $(1+x)$, x and x^2:

$$W = \begin{vmatrix} (1+x) & x & x^2 \\ 1 & 1 & 2x \\ 0 & 0 & 2 \end{vmatrix} = 2((1+x) - x) = 2 \neq 0.$$

So we conclude they are linearly independent.

2. Consider the three functions x, x^2 and $4x - 3x^2$:

$$W = \begin{vmatrix} x & x^2 & (4x - 3x^2) \\ 1 & 2x & (4 - 6x) \\ 0 & 2 & -6 \end{vmatrix} = 0.$$

So we conclude they are linearly dependent.

3. Consider the three functions e^x, e^{-x} and x. Show they are linearly independent for $x > 0$: (expand by 3rd column)

$$W = \begin{vmatrix} e^x & e^{-x} & x \\ e^x & -e^{-x} & 1 \\ e^x & e^{-x} & 0 \end{vmatrix} = x \begin{vmatrix} e^x & -e^{-x} \\ e^x & e^{-x} \end{vmatrix} - \begin{vmatrix} e^x & e^{-x} \\ e^x & e^{-x} \end{vmatrix}$$

$$= xe^{x-x}[(1) - (-1)] - 0 = 2xe^0 = 2x \neq 0 \text{ for } x > 0.$$

So we conclude they are linearly independent for $x > 0$.

Definition 3.3. A linearly independent set of functions $y_1, y_2, \cdots, y_n$ of n solutions to an nth order linear homogeneous equation is called a *Fundamental Set*.

Further, if $y_1, y_2, \cdots, y_n$ is a fundamental set then the *General Solution* is given by

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x)$$

where $c_1, c_2, \cdots, c_n$ are arbitrary constants. The general solution of the homogeneous problem is often denoted by y_h or, in our book, y_c which is called the *Complementary* Solution.

Example 3.7. The functions $y_1 = e^{-x}$, $y_2 = e^x$ form a fundamental set for the differential equation $y'' - y = 0$. To see this you can easily check the y_1 and y_2 satisfy the equation so we need to show they are linearly independent.

$$W = W(y_1, y_2) = \begin{vmatrix} e^{-x} & e^x \\ -e^{-x} & e^x \end{vmatrix} = 2 \neq 0$$

so we conclude that $y = c_1 e^{-x} + c_2 e^x$ is a general solution.

For the initial value problem

$$y'' - y = 0, \quad y(0) = 0, \quad y'(0) = 2$$

we use the general solution and the initial conditions to obtain a unique solution as follows. We have $y = c_1 e^{-x} + c_2 e^x$ which implies $y' = -c_1 e^{-x} + c_2 e^x$ so

$$\begin{aligned} 0 &= y(0) = c_1 e^{-0} + c_2 e^0 = c_1 + c_2 \\ 2 &= y'(0) = -c_1 e^{-0} + c_2 e^0 = -c_1 + c_2 \end{aligned}$$

Now we solve the 2×2 system of equations

$$\begin{aligned} c_1 + c_2 &= 0 \\ -c_1 + c_2 &= 2 \end{aligned}$$

Adding the two equations together we obtain $2c_2 = 2$ which implies $c_2 = 1$. Substituting this into the first equation we find $c_1 = -1$. Finally then we obtain the unique solution

$$y = e^x - e^{-x}.$$

Example 3.8. The functions $y_1 = x$, $y_2 = x\ln(x)$ form a fundamental set for the differential equation $x^2y'' - xy' + y = 0$. Use this to solve the initial value problem

$$x^2y'' - xy' + y = 0, \;\; y(1) = 3, \;\; y'(1) = 1.$$

The general solution is $y = c_1x + c_2x\ln(x)$ which implies $y' = c_1 + c_2(\ln(x) + 1)$ so

$$\begin{aligned} 3 &= y(1) = c_1 + c_2 0 = c_1 \\ 1 &= y'(1) = c_1 + c_2 = c_1 + c_2 \end{aligned}$$

Now we solve the 2×2 system of equations

$$\begin{aligned} c_1 &= 3 \\ c_1 + c_2 &= 1 \end{aligned}$$

Adding the two equations together we obtain $c_1 = 3$ which implies $c_2 = -2$. Finally then we obtain the unique solution

$$y = 3x - 2x\ln(x).$$

The Non-homogeneous Problem

Theorem 3.3. *Consider the non-homogeneous problem $Ly = g$ where*

$$L = (a_n(x)D^n + a_{n-1}(x)D^{n-1} + \cdots + a_1(x)D + a_0(x)).$$

If y_p is any Particular *solution and y_c is the complementary solution, then the general solution of the non-homogeneous problem is $y = y_p + y_c$.*

This follows from the following simple observation. If y_p and $\widetilde{y_p}$ are two particular solutions of the non-homogeneous problem, i.e., $Ly_p = g$ and $L\widetilde{y_p} = g$, then by the superposition principle we have

$$L(\widetilde{y_p} - y_p) = L(\widetilde{y_p}) - L(y_p) = g - g = 0,$$

i.e., $(\widetilde{y_p} - y_p)$ is a solution of the homogeneous problem. But all solutions of the homogeneous problem are contained in y_c so we must have $\widetilde{y_p} - y_p = y_c$ or, in other words, $\widetilde{y_p} = y_p + y_c$. In this way we see that every particular solution is given by finding any one particular solution and adding it to y_c.

Example 3.9. Consider the non-homogeneous IVP

$$y'' + y = 1 + x^2, \quad y(0) = -2, \quad y'(0) = 1.$$

The general solution of the homogeneous problem $y'' + y = 0$ is

$$y_c = a\cos(x) + b\sin(x)$$

and a particular solution of the non-homogeneous problem is $y_p = x^2 - 1$.

Use this information to solve the IVP. The main thing we know is that the general solution of the non-homogeneous problem is $y = y_c + y_p$ so we have

$$y = a\cos(x) + b\sin(x) + x^2 - 1.$$

To solve the IVP which we use this function and the initial conditions to find the the arbitrary constants a and b. Differentiating y we get

$$y' = -a\sin(x) + b\cos(x) + 2x.$$

Therefore from $y(0) = -1$ and $y'(0) = 1$ we have

$$a\cos(0) + b\sin(0) - 1 = -2 \text{ and } -a\sin(0) + b\cos(0) + 0 = 1$$

or

$$a = -1, \quad b = 1$$

So the unique solution to the IVP is

$$y = -\cos(x) + \sin(x) + x^2 - 1.$$

Example 3.10. Consider the non-homogeneous IVP

$$y'' - y = 1 - 2x - x^2, \quad y(0) = 1, \quad y'(0) = 4.$$

The general solution of the homogeneous problem $y'' - y = 0$ is

$$y_c = ae^x + be^{-x}$$

and a particular solution of the non-homogeneous problem is $y_p = x^2 + 2x + 1$.

Use this information to solve the IVP. The main thing we know is that the general solution of the non-homogeneous problem is $y = y_c + y_p$ so we have

$$y = ae^x + be^{-x} + x^2 + 2x + 1.$$

To solve the IVP which we use this function and the initial conditions to find the the arbitrary constants a and b. Differentiating y we get

$$y' = ae^x - be^{-x} + 2x + 2.$$

Therefore from $y(0) = 1$ and $y'(0) = 4$ we have

$$a + b + 1 = 1 \text{ and } a - b + 2 = 4$$

or

$$\begin{aligned} a + b &= 0 \\ a - b &= 2 \end{aligned}$$

If we add the two equations together the b's drop out and we have $2a = 2$ so that

$$a = 1, \quad b = -1$$

So the unique solution to the IVP is

$$y = e^x - e^{-x} + x^2 + 2x + 1.$$

Example 3.11. Consider the non-homogeneous IVP

$$y'' - 4y' + 4y = 4x + 4, \quad y(0) = 0, \quad y'(0) = 0.$$

The general solution of the homogeneous problem $y'' - 4y' + 4y = 0$ is

$$y_c = ae^{2x} + bxe^{2x}$$

and a particular solution of the non-homogeneous problem is $y_p = x + 2$.

Use this information to solve the IVP. The main thing we know is that the general solution of the non-homogeneous problem is $y = y_c + y_p$ so we have

$$y = ae^{2x} + bxe^{2x} + x + 2.$$

To solve the IVP which we use this function and the initial conditions to find the the arbitrary constants a and b. Differentiating y we get

$$y' = 2ae^{2x} + b(1 + 2x)e^{2x} + 1.$$

Therefore from $y(0) = 0$ and $y'(0) = 0$ we have

$$a + 2 = 0 \text{ and } 2a + b + 1 = 0$$

or

$$a = -2, \quad 2a + b + 1 = 0$$

which implies that $a = -2$ and $b = 3$. So the unique solution to the IVP is

$$y = -2e^{2x} + 3xe^{2x} + x + 2.$$

3.2 Reduction of Order

Suppose that y_1 is a solution to the problem

$$y'' + p(x)y' + q(x)y = 0, \tag{3}$$

i.e.,

$$y_1'' + p(x)y_1' + q(x)y_1 = 0, \tag{4}$$

Our goal is to find a second linearly independent solution y_2.

The motivation for this approach is the method of variation of parameters seen earlier in the class. We seek a solution in the form

$$y_2(x) = v(x)y_1(x).$$

Taking the derivative this implies that

$$y_2' = v'y_1 + vy_1',$$

and, taking the second derivative we have

$$y_2'' = v''y_1 + 2v'y_1' + vy_1''.$$

Substituting these expressions into (3) gives

$$\big[v''y_1 + 2v'y_1' + vy_1''\big] + p(x)\big[v'y_1 + vy_1'\big] + q(x)\big[vy_1\big] = 0.$$

Collecting the terms that multiply the expression v we get

$$v\big(y_1'' + p(x)y_1' + q(x)y_1\big)$$

which is 0 since y_1 is a solution of the homogeneous problem (see (4)) so the equation for v simplifies to

$$y_1v'' + (2y_1' + p(x)y_1)v' = 0.$$

Thus we obtain

$$v'' + \frac{(2y_1' + p(x)y_1)}{y_1} v' = 0. \tag{5}$$

Setting $w = v'$ and $\dfrac{(2y_1' + p(x)y_1)}{y_1} = 2\dfrac{y_1'}{y_1} + p(x)$ this equation reduces to the first order linear equation

$$w' + \left(2\frac{y_1'}{y_1} + p(x)\right) w = 0$$

with integrating factor

$$\mu = e^{\int \left(2\frac{y_1'}{y_1} + p(x)\right)\, dx} = e^{2\ln(y_1) + \int p(x)\, dx} = e^{\ln(y_1^2)} e^{\int p(x)\, dx} = y_1^2 e^{\int p(x)\, dx}.$$

Which gives us

$$[\mu w]' = 0 \quad \Rightarrow \quad \mu w = C$$

for an arbitrary constant C. So we end up with

$$w(x) = C\frac{1}{y_1(x)^2} e^{-\int p(x)\, dx}.$$

Therefore

$$v(x) = C \int \left(\frac{e^{-\int p(x)\, dx}}{y_1(x)^2}\right)\, dx.$$

Finally then a second linearly independent solution

$$y_2(x) = Cy_1(x) \int \left(\frac{e^{-\int p(x)\, dx}}{y_1(x)^2}\right)\, dx.$$

At this point we note that we can take any constant C we want. We usually choose it to obtain the simplest answer. In particular, it can be chosen so the constant in front is 1.

Example 3.12. The function $y_1 = e^{2x}$ is a solution of the equation

$$y'' - 4y' + 4y = 0.$$

Find a second linearly independent solution y_2.

Applying the formula from reduction of order we have

$$p(x) = -4 \;\Rightarrow\; e^{-\int p(x)\,dx} = e^{4x}$$

and we have

$$\begin{aligned} y_2 &= C y_1(x) \int \left(\frac{e^{-\int p(x)\,dx}}{y_1(x)^2} \right) dx \\ &= C e^{2x} \int \frac{e^{4x}}{(e^{2x})^2}\,dx = e^{2x} \int \,dx \\ &= C x e^{2x} \end{aligned}$$

The simplest answer would be $y_2 = xe^{2x}$ taking $C = 1$.

Example 3.13. The function $y_1 = \cos(4x)$ is a solution of the equation $y'' + 16y = 0$. Find a second linearly independent solution y_2.

Applying the formula from reduction of order we have

$$p(x) = 0 \;\Rightarrow\; e^{-\int 0\,dx} = 1$$

and we have

$$\begin{aligned} y_2 &= C y_1(x) \int \left(\frac{e^{-\int p(x)\,dx}}{y_1(x)^2} \right) dx \\ &= C \cos(4x) \int \frac{1}{(\cos(4x))^2}\,dx = \cos(4x) \int \sec^2(4x)\,dx \\ &= C \cos(4x) \frac{1}{4} \tan(4x) = C \frac{1}{4} \sin(4x). \end{aligned}$$

The simplest answer would be $y_2 = \sin(4x)$ taking $C = 4$.

Example 3.14. The function $y_1 = \ln(x)$ is a solution of the equation $xy'' + y' = 0$. Find a second linearly independent solution y_2. We must first rewrite the equation in the form $y'' + py' + qy = 0$:

$$y'' + \frac{1}{x} y' = 0.$$

Applying the formula from reduction of order we have

$$p(x) = \frac{1}{x} \;\Rightarrow\; e^{-\int p(x)\,dx} = x^{-1}$$

and we have

$$\begin{aligned}
y_2 &= Cy_1(x) \int \left(\frac{e^{-\int p(x)\,dx}}{y_1(x)^2}\right) dx \\
&= C\ln(x) \int \frac{x^{-1}}{(\ln(x))^2}\,dx = e^{2x} \int \frac{dx}{x(\ln(x))^2} \\
&= C\ln(x) \int \frac{du}{u^2} \;\; (\text{ use } \; u = \ln(x),\; du = dx/x) \\
&= C\ln(x) \int u^{-2}\,du = -\ln(x)u^{-1} = -C\ln(x)(\ln(x))^{-1} = -C
\end{aligned}$$

So the simplest answer would be $y_2 = 1$ taking $C = -1$.

Example 3.15. The function $y_1 = x^4$ is a solution of $x^2y'' - 7xy' + 16y = 0$. Find a second linearly independent solution y_2. We must first rewrite the equation in the form $y'' + py' + qy = 0$:

$$y'' - \frac{7}{x}y' + \frac{16}{x^2}y = 0.$$

Applying the formula from reduction of order we have

$$p(x) = \frac{-7}{x} \;\Rightarrow\; e^{-\int p(x)\,dx} = x^7$$

and we have

$$\begin{aligned}
y_2 &= Cy_1(x) \int \left(\frac{e^{-\int p(x)\,dx}}{y_1(x)^2}\right) dx \\
&= Cx^4 \int \frac{x^7}{(x^4)^2}\,dx = Cx^4 \int \frac{dx}{x} \\
&= Cx^4\ln(x)
\end{aligned}$$

So the simplest answer would be $y_2 = x^4\ln(x)$ by taking $C = 1$.

Example 3.16. The function $y_1 = x\sin(\ln(x))$ is a solution of $x^2y'' - xy' + 2y = 0$. Find a second linearly independent solution y_2. We must first rewrite the equation in the form $y'' + py' + qy = 0$:

$$y'' - \frac{1}{x}y' + \frac{2}{x^2}y = 0.$$

Applying the formula from reduction of order we have

$$p(x) = \frac{-1}{x} \ \Rightarrow \ e^{-\int p(x)\,dx} = x$$

and we have

$$\begin{aligned}
y_2 &= Cy_1(x) \int \left(\frac{e^{-\int p(x)\,dx}}{y_1(x)^2} \right) dx \\
&= Cx\sin(\ln(x)) \int \frac{x}{(x\sin(\ln(x)))^2}\,dx \\
&= Cx\sin(\ln(x)) \int \frac{\csc^2(\ln(x))}{x}\,dx \quad (\text{ use } \ u = \ln(x),\ du = dx/x) \\
&= Cx\sin(\ln(x)) \int \csc^2(u)\,du = Cx\sin(\ln(x))(-\cot(u)) \\
&= Cx\sin(\ln(x))(-\cot(\ln(x))) = -Cx\cos(\ln(x))
\end{aligned}$$

So the simplest answer would be $y_2 = x\cos(\ln(x))$ by taking $C = -1$.

3.3 Homogeneous Linear Constant Coefficient Equations

The Second Order Case

Consider a Second Order Homogeneous Linear Constant Coefficient Equation

$$ay'' + by' + cy = 0$$

Substituting $y = e^{rx}$ into the equation we arrive at the so-called *Characteristic Equation* $ar^2 + br + c = 0$ has roots $r_1,\ r_2$ by the quadratic equation

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

An important number is the *Discriminant*: $\Delta = b^2 - 4ac$. From College Algebra you may recall there are *Three Cases* depending on the sign of the discriminant:

1. $\Delta > 0$ Real distinct roots $r_1 \neq r_2 \;\Rightarrow\;$ (general solution) $\;y = c_1 e^{r_1 x} + c_2 e^{r_2 x}$

2. $\Delta = 0$ Real double root $r_0 = r_1 = r_2 \;\Rightarrow\;$ (general solution) $y = c_1 e^{r_0 x} + c_2 x e^{r_0 x}$

3. $\Delta < 0$ Complet roots $r = \alpha \pm i\beta \;\Rightarrow\;$ (general solution) $y = c_1 e^{\alpha x}\cos(\beta x) + c_2 e^{\alpha x}\sin(\beta x)$

Here only the first case is obvious. If we have real distinct roots r_1 and r_2 then each gives a solution $e^{r_1 x}$ and $e^{r_2 x}$ which are linearly independent so they form a fundamental set and the general solution is $y = c_1 e^{r_1 x} + c_2 e^{r_2 x}$.

In case 2, we know one solution is $e^{r_0 x}$ so we appeal to the reduction of order formula to find a second linearly independent solution y_2. In the case of a double root the equation can be written in the form $y'' - 2r_0 y' + r_0^2 y = 0$. Here so

$$p(x) = -2r_0 \;\Rightarrow\; e^{-\int p(x)\,dx} = e^{2r_0 x}$$

So we have

$$\begin{aligned} y_2 &= y_1(x) \int \left(\frac{e^{-\int p(x)\,dx}}{y_1(x)^2} \right) dx \\ &= e^{r_0 x} \int \frac{e^{2r_0 x}}{(e^{r_0 x})^2}\,dx = e^{r_0 x} \int \,dx \\ &= x e^{r_0 x} \end{aligned}$$

Therefore the general solution is $y = c_1 e^{r_0 x} + c_2 x e^{r_0 x}$ as we have above.

For case number 3 we encounter complex roots. Here we first introduce a very useful thing to remember. If a quadratic equation with real coefficients has complex roots they must be complex conjugates, i.e., $r = \alpha \pm i\beta$ where α, β are real. From this and the factor and remainder theorem (from College Algebra) we find that the characteristic equation can be written as follows:

$$\begin{aligned} 0 =& [r - (\alpha + i\beta)]\,[r - (\alpha - i\beta)] = [(r - \alpha) - i\beta]\,[(r - \alpha) + i\beta)] \\ =& (r - \alpha)^2 - (i\beta)^2 = r^2 - 2\alpha r + (\alpha^2 + \beta^2). \end{aligned}$$

This can be very useful in finding the roots and, in particular, α and β.

Another tool that is particularly useful is the famous *Euler Formula*

$$e^{i\theta} = \cos(\theta) + i\sin(\theta) \tag{6}$$

which also implies (since $\cos$ is even and $\sin$ is odd)

$$e^{-i\theta} = \cos(\theta) - i\sin(\theta) \tag{7}$$

An important side result from the Euler formulas are the following formulas. Adding the formulas (6) and (7) together and dividing by 2 we arrive at

$$\cos(\theta) = \frac{e^{i\theta} + e^{i\theta}}{2}.$$

Next we subtract the formulas (6) and (7) and divide by $2i$ to arrive at

$$\sin(\theta) = \frac{e^{i\theta} - e^{i\theta}}{2i}.$$

While we will not use the above results at this time they are nevertheless important.

Returning to the solution in the case of complex roots, since we found the roots $r = \alpha \pm i\beta$ we should be able to write the general solution as

$$y = \widetilde{c}_1 e^{(\alpha+i\beta)x} + \widetilde{c}_2 e^{(\alpha-i\beta)x} = e^{\alpha x}\left[\widetilde{c}_1 e^{i\beta x} + \widetilde{c}_2 e^{-i\beta x}\right].$$

Now we can use the Euler formulas (6), (7) to obtain

$$\begin{aligned}
y =& e^{\alpha x}\left[\widetilde{c}_1 e^{i\beta x} + \widetilde{c}_2 e^{-i\beta x}\right]\\
=& e^{\alpha x}\left[\widetilde{c}_1\{\cos(\beta x) + i\sin(\beta x)\} + \widetilde{c}_2\{\cos(\beta x) - i\sin(\beta x)\}\right]\\
=& e^{\alpha x}\left[(\widetilde{c}_1 + \widetilde{c}_2)\cos(\beta x) + (\widetilde{c}_1 - \widetilde{c}_2)i\sin(\beta x)\right]\\
=& e^{\alpha x}\left[c_1\cos(\beta x) + c_2\sin(\beta x)\right]
\end{aligned}$$

where we have set

$$c_1 := (\widetilde{c}_1 + \widetilde{c}_2), \quad c_2 := (\widetilde{c}_1 - \widetilde{c}_2)i$$

and since $\widetilde{c}_1$ and $\widetilde{c}_2$ are arbitrary constants then so also are c_1 and c_2.

Example 3.17. Consider $y'' - y' - 6y = 0$ with characteristic polynomial $r^2 - r - 6 = 0$. The discriminant is positive and quadratic factors giving two real roots, namely, $r^2 - r - 6 = (r+2)(r-3) = 0$ so $r = -2, 3$ and the general solution is $y = c_1 e^{-2x} + c_2 e^{3x}$.

Example 3.18. Consider $y'' - 4y' + 5y = 0$ with characteristic polynomial $r^2 - 4r + 5 = 0$. For this example the discriminant is negative so there are complex roots $r = \alpha \pm i\beta$. In order to find α and β we write the characteristic polynomial in the form $r^2 - 2\alpha r + \alpha^2 + \beta^2 = 0$ which gives $r^2 - 2(2)r + (2)^2 + (1)^2 = 0$ and we can read off that $\alpha = 2$ and $\beta = 1$ so the general solution is $y = c_1 e^{2x}\cos(x) + c_2 e^{2x}\sin(x)$.

Example 3.19. Consider $y'' + 8y' + 16y = 0$ with characteristic polynomial $r^2 + 8r + 16 = 0$. The discriminant is zero so there is a double root. The quadratic factors $r^2 + 8r + 16 = (r+4)^2 = 0$ so $r = -4, -4$ (a double root) and the general solution is $y = c_1 e^{-4x} + c_2 x\, e^{-4x}$.

Example 3.20. Consider the IVP $y'' + 16y = 0$ with $y(0) = 2$ and $y'(0) = -4$. The characteristic polynomial is $r^2 + 16 = 0$. The discriminant is negative so there are two complex roots $r = 4i, -4i$ and the general solution is $y = c_1 \cos(4x) + c_2 \sin(4x)$. Next we differentiate to get $y' = -4c_1 \sin(4x) + 4c_2 \cos(4x)$. Applying the first IC we get $c_1 = 2$ and applying the second IC we get $4C_2 = -4$ so that $C_2 = -1$ and the solution is $y = 2\cos(4x) - \sin(4x)$.

The Higher Order Case

This completes our discussion of the second order case. We now turn to the more general case of a homogeneous linear differential equation with constant real coefficients of order n which has the form

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = 0. \tag{8}$$

We can introduce the notation $D = \dfrac{d}{dx}$ and write the above equation as

$$P(D)y \equiv \left(a_n D^n + a_{n-1} D^{(n-1)} + \cdots + a_0\right) y = 0.$$

By the fundamental theorem of algebra we can factor $P(D)$ as

$$a_n(D-r_1)^{m_1}\,\cdots\,(D-r_k)^{m_k}\,(D^2-2\alpha_1 D+\alpha_1^2+\beta_1^2)^{p_1}\,\cdots\,(D^2-2\alpha_\ell D+\alpha_\ell^2+\beta_\ell^2)^{p_\ell},$$

where $\displaystyle\sum_{j=1}^{k} m_j + 2\sum_{j=1}^{\ell} p_j = n.$

There are two types of factors $(D-r)^k$ and $(D^2-2\alpha D+\alpha^2+\beta^2)^k$:

1. The general solution of $(D-r)^k y = 0$ is

$$y = \left(c_1 + c_2 x + \cdots + c_k x^{(k-1)}\right) e^{rx}$$

2. The general solution of $(D^2-2\alpha D+\alpha^2+\beta^2)^k y = 0$ is

$$y = \left(c_1 + c_2 x + \cdots + c_k x^{(k-1)}\right) e^{\alpha x}\cos(\beta x) + \left(d_1 + d_2 x + \cdots + d_k x^{(k-1)}\right) e^{\alpha x}\sin(\beta x).$$

Finally then the general solution of (8) contains one such term for each term in the factorization.

Rather than use D notation we can also argue as before and seek solutions of (8) in the form $y = e^{rx}$ to get a characteristic polynomial

$$a_n r^n + a_{n-1} r^{(n-1)} + \cdots + a_0 = 0.$$

In either case we find that the general solution consists of a sum of arbitrary constants $\{c_j\}_{j=1}^n$ multiplied times elements of a fundamental set, $\{y_j\}_{j=1}^n$, where each y_j has one of the following forms: x^k, $x^k e^{rx}$, $x^k e^{\alpha x}\cos(\beta x)$ or $x^k e^{\alpha x}\sin(\beta x)$. The y_j are linearly independent and the general solution is $y = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n$.

The best way to learn what to do is by working examples so let's consider some examples of higher order homogeneous problems with constant coefficients.

Example 3.21. Consider $y''' - 4y'' - 5y' = 0$ with characteristic polynomial $r^3 - 4r^2 - 5r = 0$.

This cubic polynomial factors in $r(r-5)(r+4)=0$ and we have roots $r=0,5,-4$ so the general solution is $y=c_1+c_2e^{5x}+c_3e^{-4x}$.

Example 3.22. Consider $y'''+3y''-4y'-12y=0$ with auxiliary polynomial $r^3+3r^2-4r-12=0$. We find the roots of this polynomial by factoring by grouping

$$0=r^3+3r^2-4r-12=r^2(r+3)-4(r+3)=(r+3)(r^2-4)=(r+3)(r-2)(r+2)$$

so the roots are $r=-3,2,-2$ and $y=c_1e^{-3x}+c_2e^{2x}+c_3e^{-2x}$.

The ***Rational Root Test*** which states that if $p(r)=a_nr^n+a_{n-1}r^{n-1}+\cdots+a_1r+a_0$ with integer coefficients, $r=q_1/q_2$ is a rational root in lowest terms (i.e., q_1 and q_2 are integers having no nontrivial common factors) of $p(r)=0$, then q_1 divides evenly into a_0 and q_2 divides evenly into a_n.

We also employ the factor and remainder theorem and synthetic division. Please consult a college algebra or pre-calculus book for more details.

1. The ***Factor Theorem*** states that $(r-a)$ is a factor of $p(r)$ if and only if $p(a)=0$.

2. The ***Remainder Theorem*** states that if a polynomial $p(r)$ of degree n is divided by a factor $(r-a)$ then the remainder (which is a number) $R=p(a)$. Here we have by the division algorithm

$$\frac{p(r)}{(r-a)}=q(r)+\frac{R}{(r-a)} \quad \Rightarrow \quad p(r)=(r-a)q(r)+R$$

where R is the remainder and $q(r)$ is the quotient polynomial of degree $(n-1)$.

The synthetic division form of this is

$$\begin{array}{c|cccc} & a_n & a_{n-1} & \cdots & a_0 \\ & & aa_n & \cdots & \cdots \\ \hline a & a_n & (aa_n+a_{n-1}) & \cdots & R \end{array}.$$

Here the coefficients of the quotient polynomial are in the third row.

Example 3.23. Consider $y''' - 5y'' + 3y' + 9y = 0$ with characteristic polynomial $r^3 - 5r^2 + 3r + 9 = 0$. This is a cubic polynomial and it factors but it is not obvious how. We apply the rational root test to find that the only possible rational roots are $r = \pm 1, \pm 3, \pm 9$. We try synthetic division to synthesize $(r-1)$ divided into $r^3 - 5r^2 + 3r + 9$.

$$\begin{array}{r|rrrr} & 1 & -5 & 3 & 9 \\ 1 & & 1 & -4 & -1 \\ \hline & 1 & -4 & -1 & 8 \end{array}$$

From this we see that $r = 1$ is not a root since the remainder is $R = 8$. Next we try synthetic division to synthesize $(r+1)$ divided into $r^3 - 5r^2 + 3r + 9$

$$\begin{array}{r|rrrr} & 1 & -5 & 3 & 9 \\ -1 & & -1 & 6 & -9 \\ \hline & 1 & -6 & 9 & 0 \end{array}$$

We see that $R = 0$ so that $r = -1$ is a root and also the quotient polynomial is a quadratic $q(r) = r^2 - 6r + 9$ which factors into $(r-3)^2$ and has a double root $r = 3, 3$.

So the roots in this case are $r = -1, 3, 3$ and the general solution is

$$y = c_1 e^{-x} + c_2 e^{3x} + c_3 x e^{3x}.$$

Example 3.24. Consider $y''' + 3y'' + 3y' + y = 0$ with characteristic polynomial $r^3 + 3r^2 + 3r + 1 = 0$. This is a cubic polynomial and it factors but it is not obvious how. We apply the rational root test to find that the only possible rational roots are $r = \pm 1$. We try synthetic division to compute $(r-1)$ divided into $r^3 + 3r^2 + 3r + 1$.

$$\begin{array}{r|rrrr} & 1 & 3 & 3 & 1 \\ 1 & & 1 & 4 & 7 \\ \hline & 1 & 4 & 7 & 8 \end{array}$$

From this we see that $r = 1$ is not a root since the remainder is $R = 8$. Next we try synthetic division to compute $(r+1)$ divided into $r^3 - 5r^2 + 3r + 9$

$$\begin{array}{r|rrrr} & 1 & 3 & 3 & 1 \\ -1 & & -1 & -2 & -1 \\ \hline & 1 & 2 & 1 & 0 \end{array}$$

We see that $R = 0$ so that $r = -1$ is a root and also the quotient polynomial is a quadratic $q(r) = r^2 + 2r + 1$ which factors into $(r+1)^2$ so $r = -1$ a double root $r = -1, -1$.

So the roots in this case are $r = -1, -1, -1$ and the general solution is

$$y = c_1 e^{-x} + c_2 x e^{-x} + c_3 x^2 e^{-x}.$$

Sometimes a higher order equation can be easily factored using simple algebraic techniques as the following example demonstrates

Example 3.25. Consider $y^{(4)} - 13y'' + 36y = 0$ with auxiliary polynomial $r^4 - 13r^2 + 36 = 0$. We can factor this as follows

$$r^4 - 13r^2 + 36 = (r^2 - 9)(r^2 - 4) = 0$$

The terms $(r^2 - 9) = 0$ and $(r^2 - 4) = 0$ each have real roots $r = \pm 3$ and $r = \pm 2$ so the general solution is

$$y = c_1 e^{-3x} + c_2 e^{3x} + c_3 e^{-2x} + c_4 e^{2x}.$$

Here is another similar example

Example 3.26. Consider $16y^{(4)} + 24y'' + 9y = 0$ with auxiliary polynomial $16r^4 + 24r^2 + 9 = 0$. We can write this as follows

$$(4r^2)^2 + (2)(4r^2)(3) + 3^2 = (4r^2 + 3)^2 = 0.$$

Notice this equation is 4th order so it has to have four roots. We find that $4r^2 + 3 = 0$ has roots $r = \pm\sqrt{3}/2i$ so the double roots are $0 + \sqrt{3}/2i, 0 + \sqrt{3}/2i$ and $0 - \sqrt{3}/2i, 0 - \sqrt{3}/2i$.

We obtain the general solution

$$y = (c_1 + c_2 x)\cos(\sqrt{3}/2x) + (c_3 + c_4 x)\sin(\sqrt{3}/2x).$$

Example 3.27. Consider $y''' - y' = 0$ with initial conditions $y(0) = 0$, $y'(0) = 2$, $y''(0) = 2$. To find the general solution we consider the auxiliary polynomial $r^3 - r = 0$ which factors to $r(r-1)(r+1) = 0$ with roots $r = 0, -1, 1$ and the general solution is $y = c_1 + c_2 e^{-x} + c_3 e^{x}$. Then we also need $y' = -c_2 e^{-x} + c_3 e^{x}$ and $y'' = c_2 e^{-x} + c_3 e^{x}$. Applying the ICs we get

$$\begin{aligned} c_1 + c_2 + c_3 &= 0 \\ -c_2 + c_3 &= 2 \\ c_2 + c_3 &= 2 \end{aligned}$$

Notice we can solve the last two equations for c_2 and c_3. Adding the equations together we get $2c_3 = 4$ so that $c_3 = 2$. Then from the last equation we must have $c_2 = 0$. Finally plugging in these values into the first equation we find $c_1 + 0 + 2 = 0$ so that $c_1 = -2$.

Therefore the unique solution of the IVP is

$$y = -2 + 2e^{x}.$$

Let's try one a little harder

Example 3.28. Consider $y^{(4)} + 13y'' + 36y = 0$ with initial conditions $y(0) = 0$, $y'(0) = 30$, $y''(0) = 0$, $y'''(0) = 0$. To find the general solution we consider the auxiliary polynomial $r^4 + 13r^2 + 36 = 0$. Notice that this equation cannot have any real roots. This expression factors to $(r^2 + 4)(r^2 + 9) = 0$ with roots $r = 0 + 2i, 0 - 2i, 0 + 3i, 0 - 3i$ and the general solution is $y = c_1\cos(2x) + c_2\sin(2x) + c_3\cos(3x) + c_4\sin(3x)$. Then we also need

$$y = c_1\cos(2x) + c_2\sin(2x) + c_3\cos(3x) + c_4\sin(3x)$$

$$y' = -2c_1\sin(2x) + 2c_2\cos(2x) - 3c_3\sin(3x) + 3c_4\cos(3x),$$

$$y'' = -4c_1\cos(2x) - 4c_2\sin(2x) - 9c_3\cos(3x) - 9c_4\sin(3x),$$

and

$$y''' = 8c_1\sin(2x) - 8c_2\cos(2x) + 27c_3\sin(3x) - 27c_4\cos(3x).$$

Applying the ICS we get

$$c_1 + 0c_2 + c_3 + 0c_4 = 0$$
$$0c_1 + 2c_2 + 0c_3 + 3c_4 = 30$$
$$-4c_1 + 0c_2 - 9c_3 + 0c_4 = 0$$
$$0c_1 - 8c_2 + 0c_3 - 27c_4 = 0.$$

Consider the first and third equations together

$$c_1 + c_3 = 0$$
$$-4c_1 - 9c_3 = 0$$

which gives $c_1 = c_3 = 0$. Now consider the second and forth which give

$$2c_2 + 3c_4 = 30$$
$$-8c_2 - 27c_4 = 0$$

Adding 4 times the first equation to the second, the c_2 drop out and we have $12c_4 - 27c_4 = 120$ or $-15c_4 = 120$ which gives $c_4 = -8$ and using this value in either equation we find $c_2 = 27$. Therefore the unique solution of the IVP is

$$y = 27\sin(2x) - 8\sin(3x).$$

Example 3.29. Consider $y''' + y'' - 2y = 0$ with $y(0) = 0$, $y'(0) = 3$, $y''(0) = -1$. To find the general solution we have the auxiliary polynomial $r^3 + r^2 - 2 = 0$. To find the roots of this equation we need synthetic division and the rational root test. The possible rational roots are ± 1 and ± 2. Let us try $r = 1$

$$
\begin{array}{c|cccc}
 & 1 & 1 & 0 & -2 \\
1 & & 1 & 2 & 2 \\
\hline
 & 1 & 2 & 2 & 0
\end{array}
$$

We see that the quotient polynomial is r^2+2r+2 which has complex roots $-1\pm i$ since we can write it as $r^2-2(-1)r+(-1)^2+(1)^2$. Therefore the general solution is

$$y = c_1e^x + c_2e^{-x}\cos(x) + c_3e^{-x}\sin(x).$$

To solve the initial value problem we need to find $y'(x)$ and $y''(x)$. To do this we need to use the product rule.

$$
\begin{aligned}
y' =& c_1e^x + c_2e^{-x}(-\cos(x) - \sin(x)) + c_3e^{-x}(-\sin(x) + cos(x)) \\
=& c_1e^x + (-c_2 + c_3)e^{-x}\cos(x) + (-c_2 - c_3)e^{-x}\sin(x)
\end{aligned}
$$

Next we need

$$
\begin{aligned}
y'' =& c_1e^x + (-c_2 + c_3)e^{-x}(-\cos(x) - \sin(x)) + (-c_2 - c_3))e^{-x}(-\sin(x) + cos(x)) \\
=& c_1e^x + (-2c_3)e^{-x}\cos(x) + (2c_2)e^{-x}\sin(x)
\end{aligned}
$$

Applying the initial conditions we have

$$
\begin{aligned}
&c_1 + c_2 = 0 \\
&c_1 - c_2 + c_3 = 3 \\
&c_1 - 2c_3 = -1
\end{aligned}
$$

Solving this system of 3 equations in 3 unknowns we get $c_1 = 1$, $c_2 = -1$ and $c_3 = 1$. So the unique solution of the initial value problem is $y = e^x - e^{-x}\cos(x) + e^{-x}\sin(x)$.

Example 3.30. Consider $y^{(4)} - 81y = 0$ with initial conditions $y(0) = 2$, $y'(0) = 6$, $y''(0) = 0$, $y'''(0) = 0$. To find the general solution we consider the auxiliary polynomial $r^4 - 81 = 0$. This expression factors to $(r^2 - 9)(r^2 + 9) = 0$ with roots $r = 0 + 3i, 0 - 3i, 3, -3$ and the general solution is $y = c_1 \cos(3x) + c_2 \sin(3x) + c_3 e^{-3x} + c_4 e^{3x}$. Then we also need

$$y = c_1 \cos(3x) + c_2 \sin(3x) + c_3 e^{-3x} + c_4 e^{3x}$$

$$y' = -3c_1 \sin(3x) + 3c_2 \cos(3x) - 3c_3 e^{-3x} + 3c_4 e^{3x},$$

$$y'' = -9c_1 \cos(3x) - 9c_2 \sin(3x) + 9c_3 e^{-3x} + 9c_4 e^{3x},$$

and

$$y''' = 27c_1 \sin(3x) - 27c_2 \cos(3x) - 27c_3 e^{-3x} + 27c_4 e^{3x}.$$

Applying the ICS we get

$$\begin{aligned}
&c_1 + 0c_2 + c_3 + c_4 = 2 \\
&0c_1 + 3c_2 - 3c_3 + 3c_4 = 6 \\
&-9c_1 + 0c_2 + 9c_3 + 9c_4 = 0 \\
&0c_1 - 27c_2 - 27c_3 - 27c_4 = 0
\end{aligned}$$

This simplifies to

$$\begin{aligned}
&c_1 + c_3 + c_4 = 2 \\
&3c_2 - 3c_3 + 3c_4 = 6 \\
&-9c_1 + 9c_3 + 9c_4 = 0 \\
&-27c_2 - 27c_3 + 27c_4 = 0
\end{aligned}$$

Now divide the second equation by 3, the third by 9 and the last by 27 to get

$$\begin{aligned}
&c_1 + c_3 + c_4 = 2 \\
&c_2 - c_3 + c_4 = 2 \\
&-c_1 + c_3 + c_4 = 0 \\
&-c_2 - c_3 + c_4 = 0
\end{aligned}$$

Subtract the third equation from the first and the c_3+c_4 drops out to give $2c_1 = 2$ so $c_1 = 1$.

Next in the big system above subtract the fourth equation from the second to get $2c_2 = 2$ so that $c_2 = 1$.

Plugging these values in to the big system we then have

$$\begin{aligned} 1 + c_3 + c_4 &= 2 \\ 1 - c_3 + c_4 &= 2 \\ -1 + c_3 + c_4 &= 0 \\ -1 - c_3 + c_4 &= 0 \end{aligned}$$

Lets look at the first two equations which simplify to

$$\begin{aligned} c_3 + c_4 &= 1 \\ -c_3 + c_4 &= 1 \end{aligned}$$

Adding these equations together we find $2c_4 = 2$ so $c_4 = 1$ and then this implies $c_3 = 0$. These same values satisfy the third and fourth equations above so we have $c_1 = 1$, $c_2 = 1$, $c_3 = 0$ and $c_4 = 1$.

Therefore the unique solution of the IVP is

$$y = \cos(3x) + \sin(3x) + e^{3x}.$$

Example 3.31. Consider $y'''' - 4y''' + 7y'' - 6y' + 2y = 0$ with initial conditions $y(0) = 2$, $y'(0) = 0$, $y''(0) = -3$, $y'''(0) = -6$. To find the general solution we consider the auxiliary polynomial $r^4 - 4r^3 + 7r^2 - 6r + 2 = 0$. Notice that the only possible rational roots are ± 1 and ± 2.

$$\begin{array}{c|ccccc} & 1 & -4 & 7 & -6 & 2 \\ 1 & & 1 & -3 & 4 & -2 \\ \hline & 1 & -3 & 4 & -2 & 0 \end{array}$$

Therefore $r = 1$ is a root. But it might be a double root so we try it again on the quotient polynomial

$$\begin{array}{c|cccc} & 1 & -3 & 4 & -2 \\ 1 & & 1 & -2 & 2 \\ \hline & 1 & -2 & 2 & 0 \end{array}$$

And, we see that $r = 1$ is a root once again. Therefore $r = 1$ is a double root. At this point the quotient polynomial is quadratic so we only need to find the roots of $r^2 - 2r + 2$ and a quick check of the discriminant shows it has complex roots. Namely we have $r^2 - 2(1)r + (1)^2 + (1)^2$ which implies that $r = 1 \pm i$. Finally then the 4 roots are $1, 1, 1 \pm i$. Then we can write the general solution as

$$y = c_1 e^x + c_2 x e^x + c_3 e^x \cos(x) + c_4 e^x \sin(x).$$

We need to find the constants c_1, c_2, c_3, c_4 so that the initial conditions are satisfied. This requires us to compute y', y'' and y'''. We have

$$\begin{aligned} y' &= c_1 e^x + c_2(1+x)e^x + c_3 e^x(\cos(x) - \sin(x)) + c_4 e^x(\sin(x) + \cos(x)) \\ &= (c_1 + c_2 + c_2 x)e^x + (c_3 + c_4)e^x \cos(x) + (-c_3 + c_4)e^x \sin(x). \end{aligned}$$

$$\begin{aligned} y'' &= [c_2 + c_1 + c_2 + c_2 x]e^x + (c_3 + c_4)e^x(\cos(x) - \sin(x)) + (-c_3 + c_4)e^x(\sin(x) + \cos(x)) \\ &= (c_1 + 2c_2 + c_2 x)e^x + (2c_4)e^x \cos(x)(-2c_3)e^x \sin(x). \end{aligned}$$

$$\begin{aligned} y''' &= [c_1 + 3c_2 + c_2 x]e^x + (2c_4)e^x(\cos(x) - \sin(x)) + (-2c_3)e^x(\sin(x) + \cos(x)) \\ &= [c_1 + 3c_2 + c_2 x]e^x + (-2c_3 + 2c_4)e^x \cos(x) + (-2c_3 - 2c_4)e^x \sin(x). \end{aligned}$$

From the initial conditions we have

$$\begin{aligned} &c_1 + c_3 = 2 \\ &c_1 + c_2 + c_3 + c_4 = 0 \\ &c_1 + 2c_2 + 2c_4 = -3 \end{aligned}$$

$$c_1 + 3c_2 - 2c_3 + 2c_4 = -6$$

1. From the first equation we have $c_3 = 2 - c_1$
2. Replacing c_3 by $2 - c_1$ in the second equation we have $c_2 + c_4 = -2$ which implies that $c_4 = -2 - c_2$.
3. Substituting $c_4 = -2 - c_2$ into the third equation we have $\boxed{c_1 = 1}$.
4. Using $c_1 = 1$, $c_3 = 2 - c_1$ and $c_4 = -2 - c_2$ in the fourth equation we have $\boxed{c_2 = -1}$
5. But then by item 1 above we have $\boxed{c_3 = 1}$.
6. And, finally, by item 2 we have $\boxed{c_4 = -1}$.

Therefore the unique solution of the IVP is

$$y = e^x - xe^x + e^x \cos(x) - e^x \sin(x).$$

3.4 Method of Undetermined Coefficients

Non-Homogeneous Problem:

We now turn to the hardest part of Chapter 3, finding the general solution to the non-homogeneous problem:

$$Ly = (a_n D^n + a_{n-1} D^{n-1} + \cdots + a_0)y = f(x) \tag{9}$$

As we have already mentioned, the general solution is obtained as $y = y_c + y_p$ where

1. y_c is the general solution of the homogeneous (or complementary) problem, i.e. $y_c = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n$ where $y_1, \cdots, y_n$ are n linearly independent solutions of

$$Ly = (a_n D^n + a_{n-1} D^{n-1} + \cdots + a_0)y = 0$$

with the <u>*Characteristic Polynomial*</u>

$$a_n r^n + a_{n-1} r^{n-1} + \cdots + a_0 = 0 \tag{10}$$

2. y_p is (any) particular solution of the non-homogeneous problem (9).

The main problem then is to find y_p.

Remark 3.3. We will be mostly concerned with the general solution in case the left hand side is a second order equation

$$ay'' + by' + cy = f(x).$$

Method of Undetermined Coefficients

The method of undetermined coefficients is only applicable if the right hand side is a sum of terms of the following form

$$p(x), \quad p(x)e^{ax}, \quad p(x)e^{\alpha x}\cos(\beta x), \quad p(x)e^{\alpha x}\sin(\beta x) \tag{11}$$

where we denote by $p(x) = c_m x^m + c_{m-1}x^{m-1} + \cdots + c_0$ a general polynomial of degree m. For a right hand side function consisting of a sum of terms like these, y_p will be found as a sum of such terms. Each of the individual terms are computed using the following:

BOX 1:

$$ay'' + by' + cy = p(x)e^{r_0 x} \quad \Rightarrow \quad y_p = x^s(A_m x^m + \cdots + A_1 x + A_0)e^{r_0 x}$$

1. $s = 0$ if r_0 is not a root of $ar^2 + br + c = 0$.
2. $s = \ell$ if r_0 is a root ℓ times of $ar^2 + br + c = 0$ (here $\ell = 1$ or 2).

N.B. The above case includes the case $r_0 = 0$ in which case the right side is $p(x)$.

BOX 2:

$$ay'' + by' + cy = \begin{cases} p(x)e^{\alpha x}\cos(\beta x) \\ \quad\text{or} \\ p(x)e^{\alpha x}\sin(\beta x) \end{cases} \quad \Rightarrow$$

$$y_p = x^s(A_m x^m + \cdots + A_1 x + A_0)e^{\alpha x}\cos(\beta x) + x^s(B_m x^m + \cdots + B_1 x + B_0)e^{\alpha x}\sin(\beta x)$$

1. $s = 0$ if $r_0 = \alpha + i\beta$ is not a root of $ar^2 + br + c = 0$.
2. $s = 1$ if $r_0 = \alpha + i\beta$ is a root of $ar^2 + br + c = 0$.

Remark 3.4. It can happen that the function $f(x)$ on the right hand side is a sum of several functions each of which must be handled separately. For example

$$f(x) = f_1(x) + f_2(x) + \cdots + f_n(x)$$

where each $f_j(x)$ is of the form described in BOX 1 or BOX 2 but with different r_0 or α and β. Notice that if one of the terms is a polynomial, e.g., $3x^3 + 2x^2 + x + 1$, then this is to be considered as a single function corresponding to $r_0 = 0$ and not several different functions.

So let us consider $Ly = ay'' + by' + cy$ and the associated non-homogeneous problem

$$Ly = f_1(x) + f_2(x) + \cdots + f_n(x)$$

To find y_p for a situation like this we simply find n particular solutions y_{p_j} satisfying $Ly_{p_j} = f_j$ and add them together. Namely we have

$$y_p = y_{p_1} + y_{p_2} \cdots + y_{p_n}.$$

The reason this works is that the problem is linear:

$$Ly_p = L(y_{p_1} + \cdots + y_{p_n}) = L(y_{p_1}) + \cdots + L(y_{p_n}) = f_1 + \cdots + f_n = f.$$

In the following examples you are asked to find a candidate for a particular solution. This means we give the form of the particular solution but do not find the values of the coefficients themselves.

1. $y''-2y'+2y=2e^x\cos(x) \quad \Rightarrow$ For the homogenous problem we have $y''-2y'+2y=0$ $\Rightarrow r^2-2r+2=0 \Rightarrow\ r=1\pm i$ so we have $y_c=c_1e^x\cos(x)+c_2e^x\sin(x)$. The right hand side has $p(x)=2$ (a polynomial of degree 0, i.e., a constant), $r_0=1+i$ which is a root once of the characteristic polynomial. So we look at BOX 2 with $s=1$ we have

$$y_p=Axe^x\sin(x)+Bxe^x\cos(t)$$

2. $y''-2y'+y=2e^x \quad \Rightarrow$ For the homogenous problem we have $y''-2y'+y=0\Rightarrow$ $r^2-2r+1=0\Rightarrow\ r=1,1$ is a double root. So we have $y_c=c_1e^x+c_2xe^x$. The right hand side has $p(x)=2$ (a polynomial of degree 0, i.e., a constant), $r_0=1$ which is a root twice of the characteristic polynomial. So we look at BOX 1 with $s=2$ and we have

$$y_p=Ax^2e^x$$

3. $y''-4y'+3y=x^2+x-1+\sin(x) \quad \Rightarrow$

   For the homogenous problem we have $y''-4y'+3y=0\Rightarrow r^2-4r+3=0\Rightarrow\ r=3,4$ so we have $y_c=c_1e^{3x}+c_2e^{4x}$. Following the discussion in Remark 3.4 we see that the right hand side has two parts:

   (a) For the first we have $p(x)=x^2+x-1$ (a polynomial of degree 2, i.e., a quadratic), and $r_0=0$ which is NOT a root of the characteristic polynomial. So we look at BOX 1 with $s=0$ and we have $y_{p_1}=(Ax^2+Bx+C)$.

   (b) For the second part we have $p(x)=1$ (a polynomial of degree 0, i.e., a constant), and $r_0=0+i$. We note that r_0 is not a root of the characteristic polynomial so $s=0$ and we have $y_{p_2}=D\sin(x)+E\cos(x)$.

   Adding these together we arrive at

$$y_p=(Ax^2+Bx+C)+(D\sin(x)+E\cos(x))$$

4. $y''+9y=\sin(2x) \ \Rightarrow\ \ y_p=A\sin(2x)+B\cos(2x)$

5. $y'' - 3y' + 2y = e^x \quad \Rightarrow \quad y_p = Axe^x$

6. $y'' - y' = x + 1 \quad \Rightarrow \quad y_p = Ax^2 + Bx$

Let us turn now to the problem of actually finding a particular solution. We will present a few simple examples.

Example 3.32. Find the general solution for $y'' + 3y' + 2y = 6$.

1. First we solve the homogeneous problem $y'' + 3y' + 2y = 0$ by finding the roots of the characteristic equation $r^2 + 3r + 2 = 0$ which gives $(r + 2)(r + 1) = 0$ which implies $r = -1$ $r = -2$ so we have $y_c = c_1 e^{-x} + c_2 e^{-2x}$.

2. Next we need to find y_p so we first need to find a candidate for a particular solution. The function on the right hand side is from BOX 1 with $m = 0$ (a polynomial of degree zero) and $r_0 = 0$ which is not a root of the characteristic equation. So we have $y_p = Ae^{0x} = A$. To find y_p we now need to find A and we do this by plugging this y_p into the given equation and solve for A.

   We have $y_p = A$, $y_p' = 0$, $y_p'' = 0$ so we obtain

$$(0) + 3(0) + 2(A) = 6.$$

   This gives $2A = 6$ which implies $A = 3$. So $y_p = 3$.

3. Finally then the general solution for this problem is

$$y = y_c + y_p = c_1 e^{-x} + c_2 e^{-2x} + 3.$$

Example 3.33. Find the general solution for $y'' + 3y' + 2y = 40e^{3x}$.

1. First we solve the homogeneous problem $y'' + 3y' + 2y = 0$ by finding the roots of the characteristic equation $r^2 + 3r + 2 = 0$ which gives $(r + 2)(r + 1) = 0$ which implies $r = -1$ $r = -2$ so we have $y_c = c_1 e^{-x} + c_2 e^{-2x}$.

2. Next we need to find y_p so we first need to find a candidate for a particular solution. The function on the right hand side is from BOX 1 with $m = 0$ (a polynomial of

degree zero) and $r_0 = 3$ which is not a root of the characteristic equation. So we have $y_p = Ae^{3x}$. To find y_p we now need to find A and we do this by plugging this y_p into the given equation and solve for A.

We have $y_p = Ae^{3x}$, $y_p' = 3Ae^{3x}$, $y_p'' = 9Ae^{3x}$ so we obtain

$$(9Ae^{3x}) + 3(3Ae^{3x}) + 2(Ae^{3x}) = 40Ae^{3x}.$$

This gives $20A = 40$ which implies $A = 2$. So $y_p = 2e^{3x}$.

3. Finally then the general solution for this problem is

$$y = y_c + y_p = c_1e^{-x} + c_2e^{-2x} + 2e^{3x}.$$

Example 3.34. Find the general solution for $y'' - y' = 4x$.

1. First we solve the homogeneous problem $y'' - y' = 0$ by finding the roots of the characteristic equation $r^2 - r = 0$ which gives $r(r-1) = 0$ which implies $r = 0$, $r = 1$ so we have $y_c = c_1 + c_2e^x$.

2. Next we need to find y_p so we first need to find a candidate for a particular solution. The function on the right hand side is from BOX 1 with $m = 1$ (a polynomial of degree one) and $r_0 = 0$ which is a root of the characteristic equation once. So we have $y_p = x(Ax + B)$. To find y_p we now need to find A and we do this by plugging this y_p into the given equation and solve for A and B.

   We have $y_p = Ax^2 + Bx$, $y_p' = 2Ax + B$, $y_p'' = 2A$ so we obtain

$$(2A) - (2Ax + B) = 4x.$$

   This gives $2A - B = 0$ and $-2A = 4$ which implies $A = -2$ and $B = -4$. So $y_p = -2x^2 - 4x$.

3. Finally then the general solution for this problem is

$$y = y_c + y_p = c_1 + c_2e^x - 2x^2 - 4x.$$

Example 3.35.

Find the general solution for $y'' + 3y' + 2y = 10\sin(x)$.

1. First we solve the homogeneous problem $y'' + 3y' + 2y = 0$ by finding the roots of the characteristic equation $r^2 + 3r + 2 = 0$ which gives $(r+2)(r+1) = 0$ which implies $r = -1$ $r = -2$ so we have $y_c = c_1 e^{-x} + c_2 e^{-2x}$.

2. Next we need to find y_p so we first need to find a candidate for a particular solution. The function on the right hand side is from BOX 2 with $m = 0$ (a polynomial of degree zero) and $r_0 = 0 + i$ which is not a root of the characteristic equation. So we have $y_p = A\cos(x) + B\sin(x)$. To find y_p we now need to find A and B which we do by plugging our candidate for y_p into the given equation and solve for A and B.

   We have $y_p = A\cos(x) + B\sin(x)$, $y_p' = -A\sin(x) + B\cos(x)$, $y_p'' = -A\cos(x) - B\sin(x)$ so we obtain

$$(-A\cos(x) - B\sin(x)) + 3(-A\sin(x) + B\cos(x)) + 2(A\cos(x) + B\sin(x)) = \sin(x).$$

   Now collect the sine and cosine terms on each side of the equation.

$$(A + 3B)\cos(x) + (-3A + B)\sin(x) = \sin(x).$$

   Equating the like terms on each side we find

$$A + 3B = 0$$
$$-3A + B = 10$$

   Taking 3 times the first equation added to the second we get $10B = 10$ which implies $B = 1$. with $B = 1$ in the first equation we get $A = -3$ so we have $y_p = -3\cos(x) + \sin(x)$.

3. The general solution for this problem is

$$y = y_c + y_p = c_1 e^{-x} + c_2 e^{-2x} - 3\cos(x) + \sin(x).$$

4. Consider $y'' + y' - 2y = 18xe^x - 4x \quad \Rightarrow$ For the homogenous problem we have $y'' + y' - 2y = 0 \Rightarrow r^2 + r - 2 = 0 \Rightarrow \; r = 1, -2$. So we have $y_c = c_1 e^x + c_2 e^{-2x}$. Again following the discussion in Remark 3.4 we see that the right hand side has two parts:

   (a) For the first we have $p(x) = 18xe^x$ (a polynomial of degree 1), and $r_0 = 1$ which is a root once of the characteristic polynomial. So we look at BOX 1 with $s = 1$ and we have $y_{p_1} = x(Ax + B)e^x$.

   $$y'_{p_1} = (Ax^2 + (2A + B)x + B)e^x, \;\; y''_{p_1} = (Ax^2 + (4A + B)x + (2A + 2B))e^x.$$

   Substituting these into the equation and dividing both sides by e^x gives

   $$(Ax^2 + (4A + B)x + (2A + 2B)) + (Ax^2 + (2A + B)x + B) - 2(Ax^2 + Bx) = 18x$$

   Notice that the x^2 terms all cancel out and we have

   $$6Ax = 18x \quad \Rightarrow \quad A = 3.$$

   $$2A + 3B = 0 \quad \Rightarrow \quad B = -2.$$

   So we have $y_{p_1} = x(3x - 2)e^x$.

   (b) For the second part we have $p(x) = -4x$ (a polynomial of degree 1), and $r_0 = 0$. We note that r_0 is not a root of the characteristic polynomial so $s = 0$ and, we look at BOX 1, which gives $y_{p_2} = Cx + D$.

   $$y'_{p_2} = C, \;\; y''_{p_2} = 0$$

   so we have

   $$C - 2(Cx + D) = -4x$$

   which implies

   $$-2C = -4 \;\Rightarrow\; C = 2, \text{ and } C - 2D = 0 => D = 1$$

so that $y_{p_2} = 2x + 1$

Adding these together we arrive at

$$y_p = y_{p_1} + y_{p_2} = x(Ax + B)e^x + Cx + D = x(3x - 2)e^x + 2x + 1.$$

Example 3.36. Find the general solution for $y'' + y = x^3$. Then solve the IVP $y(0) = 2$ and $y'(0) = -3$.

1. First we solve the homogeneous problem $y'' + y = 0$ by finding the roots of the characteristic equation $r^2 + 1 = 0$ which gives $r = 0 \pm i$ so we have $y_c = c_1 \cos(x) + c_2 \sin(x)$.

2. Next we need to find y_p so we first need to find a candidate for a particular solution. The function on the right hand side is from BOX 1 with $m = 3$ (a polynomial of degree 3) and $r_0 = 0$ which is not a root of the characteristic equation. So we have $y_p = (Ax^3 + Bx^2 + Cx + D)$. To find y_p we now need to find A and we do this by plugging this y_p into the given equation and solve for A.

   We have $y_p = (Ax^3 + Bx^2 + Cx + D)$, $y_p' = (3Ax^2 + 2Bx + C)$, $y_p'' = (6Ax + 2B)$ so we obtain

   $$(6Ax + 2B) + (Ax^3 + Bx^2 + Cx + D) = x^3.$$

   This immediately gives $A = 1$ and $B = 0$. Then we have $6A + C = 0$ and $D + 2B = 0$ so we have $C = -6$ and $D = 0$. So then we have $y_p = x^3 - 6x$.

3. Then the general solution for this problem is

   $$y = y_c + y_p = c_1 \cos(x) + c_2 \sin(x) + x^3 - 6x.$$

4. For the IVP we have $y(0) = 2$ and $y'(0) = -3$

   $$y = c_1 \cos(x) + c_2 \sin(x) + x^3 - 6x, \quad \Rightarrow \quad c_1 = 2,$$

   $$y' = -c_1 \sin(x) + c_2 \cos(x) - 6, \quad \Rightarrow \quad c_2 - 6 = -3, \quad \Rightarrow \quad c_2 = 3.$$

Finally we have

$$y = 2\cos(x) + 3\sin(x) + x^3 - 6x.$$

Example 3.37. Find the general solution for $y'' - 2y' + 2y = 2x$. Then solve the IVP $y(0) = 0$ and $y'(0) = 0$.

1. First we solve the homogeneous problem $y'' - 2y' + 2y = 0$ by finding the roots of the characteristic equation $r^2 - 2r + 2 = 0$ which gives $r = 1 \pm i$ so we have $y_c = c_1 e^x \cos(x) + c_2 e^x \sin(x)$.

2. Next we need to find y_p so we first need to find a candidate for a particular solution. The function on the right hand side is from BOX 1 with $m = 1$ (a polynomial of degree 3) and $r_0 = 0$ which is not a root of the characteristic equation. So we have $y_p = (Ax + B)$. To find y_p we now need to find A and we do this by plugging this y_p into the given equation and solve for A.

   We have $y_p = (A + B)$, $y_p' = A$, $y_p'' = 0$ so we obtain

   $$0 - 2A + 2(Ax + B) = 2x.$$

   This immediately gives $A = 1$ and $B = 1$. So then we have $y_p = x + 1$.

3. Then the general solution for this problem is

   $$y = y_c + y_p = c_1 e^x \cos(x) + c_2 e^x \sin(x) + x + 1.$$

4. For the IVP we have $y(0) = 2$ and $y'(0) = -3$

   $$y = c_1 e^x \cos(x) + c_2 e^x \sin(x) + x + 1, \;\Rightarrow\; c_1 + 1 = 0, \;\Rightarrow\; c_1 = -1,$$

   $$y' = -c_1 e^x(\cos(x) - \sin(x)) + c_2 e^x(\sin(x) + \cos(x)) + 1, \;\Rightarrow\; (c_1 + c_2) + 1 = 0, \;\Rightarrow\; c_2 = 0.$$

   Finally we have

   $$y = -e^x \cos(x) + x + 1.$$

3.5 Variation of Parameters

In this section we consider a second order homogeneous problem (not necessarily constant coefficient).

The general second order linear equation has the form

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = g(x).$$

Under the assumption that $a_n(x)$ is not ever zero, we can divide by $a_2(x)$ and obtain the required form for the following computations

$$y'' + p(x)y' + q(x)y = f(x). \tag{12}$$

Suppose that y_1 and y_2 form a fundamental set for the homogenous problem

$$y'' + p(x)y' + q(x)y = 0$$

so that the the complementary solution is $y_c = c_1y_1 + c_2y_2$.

Our goal now is to find a particular solution y_p. In the method of *Variation of Parameters* we seek a particular solution by "varying" the two constants in the general solution of the homogeneous problem. This is a bit vague but the general idea is this. We seek a particular solution in the form

$$y_p = uy_1 + vy_2 \tag{13}$$

for some unknown (to be determined) functions u and v.

$$y_p(x) = -y_1(x)\int \frac{y_2(x)f(x)}{W(x)}\,dx + y_2(x)\int \frac{y_1(x)f(x)}{W(x)}\,dx, \quad W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \tag{14}$$

To obtain this formula we proceed by substituting $y_p = uy_1 + vy_2$ into the equation (12) and then solving for u and v as follows:

$$y_p = uy_1 + vy_2 \quad \Rightarrow \quad y_p' = uy_1' + u'y_1 + vy_2' + v'y_2.$$

At this point we make an assumption that

$$u'y_1 + v'y_2 = 0. \tag{15}$$

There is nothing wrong with making such an assumption as long as we end up finding u and v for which the assumption holds. With this assumption our formula for y_p' simplifies to

$$y_p' = uy_1' + vy_2' \tag{16}$$

which can differentiate again

$$y_p'' = (uy_1' + vy_2')' = uy_1'' + u'y_1' + vy_2'' + v'y_2'. \tag{17}$$

We now substitute the right hand side of (13) for y_p, the rhs of (16) for y_p' and the rhs of (17) for y_p'' into the equation (12). This gives

$$\begin{aligned}
f(x) =& y'' + p(x)y' + q(x)y \\
=& (uy_1'' + u'y_1' + vy_2'' + v'y_2') + p(x)(uy_1' + vy_2') + q(x)(uy_1 + vy_2) \\
=& u(y_1'' + p(x)y_1' + q(x)y_1) + v(y_2'' + p(x)y_2' + q(x)y_2) + (u'y_1' + v'y_2') \\
=& u(0) + v(0) + (u'y_1' + v'y_2') \\
=& (u'y_1' + v'y_2').
\end{aligned}$$

So we end up with two equation in the two unknowns u', v'.

$$\begin{aligned}
u'y_1 + v'y_2 &= 0 \\
u'y_1' + v'y_2' &= f
\end{aligned} .$$

This system can be solved using *Cramer's rule* (see any college algebra book). The system is solvable due to the fact that the Wronskian of y_1 and y_2 is not zero.

$$W(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix}$$

and we get

$$u' = \frac{\begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix}}{W(x)} = \frac{-y_2(x)f(x)}{W(x)},$$

and

$$v' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}}{W(x)} = \frac{y_1(x)f(x)}{W(x)}.$$

Integrating these results we arrive at

$$u = \int \frac{-y_2(x)f(x)}{W(x)}\,dx, \qquad v = \int \frac{y_1(x)f(x)}{W(x)}\,dx \tag{18}$$

and we immediately arrive at the formula (14).

Example 3.38. Consider $y'' + y = \sec(x)$. The homogeneous problem $y'' + y = 0$ has solution $y_c = c_1\cos(x) + c_2\sin(x)$ so we set $y_1 = \cos(x)$ and $y_2 = \sin(x)$.

$$W(x) = \begin{vmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{vmatrix} = \cos^2(x) + \sin^2(x) = 1.$$

$$u' = \frac{-\sin(x)\sec(x)}{1} = \frac{-\sin(x)}{\cos(x)}, \quad \Rightarrow \quad u = -\int \frac{\sin(x)}{\cos(x)}\,dx = \ln(\cos(x)).$$

$$v' = \frac{\cos(x)\sec(x)}{1} = 1, \quad \Rightarrow \quad v = \int 1\,dx = x.$$

So we have

$$y_p = \cos(x)\,\ln(\cos(x)) + x\sin(x).$$

Example 3.39. Consider $y'' - y = 1/x$. The homogeneous problem $y'' - y = 0$ has $r^2 - 1 = 0$ so $r = \pm 1$. A fundamental set of solutions for the homogeneous problem is $y_1 = e^{-x}$ and $y_2 = e^x$ and the solution $y_c = c_1e^{-x} + c_2e^x$.

$$W(x) = \begin{vmatrix} e^{-x} & e^x \\ -e^{-x} & e^x \end{vmatrix} = 2.$$

$$u' = -\frac{e^x}{2x}, \quad \Rightarrow \quad u = -\int \frac{e^x}{2x}\,dx.$$

$$v' = \frac{e^{-x}}{2x}, \quad \Rightarrow \quad v = \int \frac{e^{-x}}{2x}\,dx.$$

The point of this exercise is that the integrals

$$\int \frac{e^x}{2x}\,dx \quad \text{and} \quad \int \frac{e^{-x}}{2x}\,dx$$

cannot be computed in closed form. In other words you cannot compute these integrals using any methods from calculus. So the answer has to be given in this form

$$y_p = -e^{-x}\int \frac{e^x}{2x}\,dx + e^x \int \frac{e^{-x}}{2x}\,dx.$$

Example 3.40. Consider $y'' - 2y' + y = 6xe^x$. The homogeneous problem $y'' - 2y' + y = 0$ has $r^2 - 2r + 1 = 0$ so $r = 1, 1$ (a double root). A fundamental set of solutions for the homogeneous problem is $y_1 = e^x$ and $y_2 = xe^x$ and the solution $y_c = c_1 e^x + c_2 x e^x$.

$$W(x) = \begin{vmatrix} e^x & xe^x \\ e^x & (1+x)e^x \end{vmatrix} = e^{2x}.$$

$$u' = -\frac{xe^x 6xe^x}{e^{2x}}, \quad \Rightarrow \quad u = -\int 6x^2\,dx = -2x^3.$$

$$v' = \frac{e^x 6xe^x}{e^{2x}}, \quad \Rightarrow \quad v = \int 6x\,dx = 3x^2.$$

$$y_p = (-2x^3)e^x + (3x^2)xe^x = x^3 e^x.$$

Example 3.41. Consider $y'' + y = 2\sin(x)$. The homogeneous problem $y'' + y = 0$ has solution $y_c = c_1\cos(x) + c_2\sin(x)$ so we set $y_1 = \cos(x)$ and $y_2 = \sin(x)$.

$$W(x) = \begin{vmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{vmatrix} = \cos^2(x) + \sin^2(x) = 1.$$

$$u' = -2\frac{\sin(x)\sin(x)}{1} = -2\sin^2(x),$$

$$u = -2\int \sin^2(x)\,dx = \frac{2}{2}\int (1-\cos(2x))\,dx = -x + \frac{1}{2}\sin(2x).$$

$$v' = 2\frac{\cos(x)\sin(x)}{1} = 2\sin(x)\cos(x), \quad \Rightarrow \quad v = \int 2\sin(x)\cos(x)\,dx = \sin^2(x).$$

So we have

$$y_p = (-x + \frac{1}{2}\sin(2x))\cos(x) + \sin^3(x) = -x\cos(x) + \sin(x).$$

Notice that $\sin(x)$ is part of y_c so we could take $y_p = -x\cos(x)$.

Example 3.42. Consider $y'' + y = \tan(x)$. The homogeneous problem $y'' + y = 0$ has solution $y_c = c_1\cos(x) + c_2\sin(x)$ so we set $y_1 = \cos(x)$ and $y_2 = \sin(x)$.

$$W(x) = \begin{vmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{vmatrix} = \cos^2(x) + \sin^2(x) = 1.$$

$$y_p = -\cos(x)\int \frac{\sin(x)\tan(x)}{1}\,dx + \sin(x)\int \frac{\cos(x)\tan(x)}{1}\,dx.$$

$$\int \frac{\sin(x)\tan(x)}{1}\,dx = \int \frac{\sin^2(x)}{\cos(x)}\,dx \int \frac{(1-\cos^2(x))}{\cos(x)}\,dx$$
$$= \int (\sec(x) - \cos(x))\,dx = \ln(|\sec(x) + \tan(x)|) - \sin(x).$$

$$\int \frac{\cos(x)\tan(x)}{1}\,dx = \int \sin(x)\,dx = -\cos(x).$$

So we have

$$y_p = -(\ln(|\sec(x) + \tan(x)|) - \sin(x))\cos(x) - \sin(x)\cos(x)$$
$$= -\ln(|\sec(x) + \tan(x)|)\cos(x).$$

Example 3.43. Consider $y''-4y'+4y=10e^{2x}$. The homogeneous problem $y''-4y'+4y=0$ has $r^2-4r+4=0$ so $r=2,2$ (a double root). A fundamental set of solutions for the homogeneous problem is $y_1=e^{2x}$ and $y_2=xe^{2x}$ and the solution $y_c=c_1e^{2x}+c_2xe^{2x}$.

$$W(x)=\begin{vmatrix} e^{2x} & xe^{2x} \\ 2e^{2x} & (1+2x)e^{2x} \end{vmatrix}=e^{4x}.$$

$$y_p=-e^{2x}\int\frac{xe^{2x}10e^{2x}}{e^{4x}}\,dx+xe^{2x}\int\frac{e^{2x}10e^{2x}}{e^{4x}}\,dx.$$

$$\int\frac{xe^{2x}10e^{2x}}{e^{4x}}\,dx=10\int x\,dx=5x^2.$$

$$\int\frac{e^{2x}10e^{2x}}{e^{4x}}\,dx=10\int 1\,dx=10x$$

So we have

$$y_p=-5x^2e^{2x}+10x^2e^{2x}=5x^2e^{2x}.$$

In the next example we compare the use of undetermined coefficients and variation of parameters.

Example 3.44. Consider $y''-y=2x+4$. The homogeneous problem $y''-y=0$ has $r^2-1=0$ so $r=-1,1$. A fundamental set of solutions for the homogeneous problem is $y_1=e^{-x}$ and $y_2=e^{x}$ and the solution $y_c=c_1e^{-x}+c_2e^{x}$.

$$W(x)=\begin{vmatrix} e^{-x} & e^{x} \\ -e^{-x} & e^{x} \end{vmatrix}=2.$$

$$y_p=-e^{-x}\int\frac{e^{x}(2x+4)}{2}\,dx+e^{x}\int\frac{e^{-x}(2x+4)}{2}\,dx.$$

$$\int\frac{e^{x}(2x+4)}{2}\,dx=\int e^{x}(x+2)\,dx$$

$$\int\frac{e^{-x}(2x+4)}{2}\,dx=\int e^{-x}(x+2)\,dx.$$

We will compute both of these integrals at once using integration by parts. With $k=\pm1$

we consider

$$\int e^{kx}(x+1)\,dx = \int \left(\frac{e^{kx}}{k}\right)'(x+2)\,dx$$
$$= \frac{e^{kx}}{k}(x+1) - \int \frac{e^{kx}}{k}\,dx = \frac{(x+2)e^{kx}}{k} - \frac{e^{kx}}{k^2}.$$

So we have

$$y_p = -e^{-x}[(x+2)e^{x} - e^{x}] + e^{x}[-(x+2)e^{-x} - e^{-x}] = -2(x+2).$$

3.6 Euler - Cauchy Equations

So far in this chapter almost all of our work has been applied to constant coefficient equations. We now turn to a class of problems that are not constant coefficient but can be handled using those methods after a substitution. We consider the so-called Euler-Cauchy Equations

$$ax^2y'' + bxy' + cy = 0 \quad \text{for} \quad x \neq 0. \tag{19}$$

One simple approach to studying these problems is to look for solutions in the form $y = x^r$. In this case we have $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Plugging these into the equation (19) we have

$$0 = ax^2\left[r(r-1)x^{r-2}\right] + bx\left[rx^{r-1}\right] + c\left[x^r\right] = (ar(r-1) + br + c)x^r.$$

Since $x \neq 0$ we can divide by x to get something like a "characteristic polynomial"

$$ar^2 + (b-a)r + c = 0 \tag{20}$$

This equation has roots $r_1,\ r_2$ just like the constant coefficient case and there are cases:

1. Real distinct roots $r_1 \neq r_2 \;\Rightarrow\;$ (general solution) $y = c_1x^{r_1} + c_2x^{r_2}$

2. Real double root $r_0 = r_1 = r_2 \;\Rightarrow\;$ (general solution) $y = c_1x^{r_0} + c_2\ln(x)x^{r_0}$

3. Complet roots $r = \alpha \pm i\beta \;\Rightarrow\;$ (general solution) $y = c_1x^{\alpha}\cos(\beta\ln(x)) + c_2x^{\alpha}\sin(\beta\ln(x))$

Only the first case is obvious. In the case of a double root or complex roots it is perhaps easier to see the big picture by taking a slightly different approach. Let us consider a change of variables that will transform the problem (19) to a problem with constant coefficients. We set $x = e^t$ which is equivalent to $\Rightarrow\ t = \ln(x)$. Using this change of variables we have

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{1}{x}\frac{dy}{dt},$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dt}\right) = -\frac{1}{x^2}\frac{dy}{dt} + \frac{1}{x^2}\frac{d^2y}{dt^2}.$$

Substituting these expressions into the differential equation (19) we arrive at

$$ax^2\left[\frac{1}{x^2}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right)\right] + bx\left[\frac{1}{x}\frac{dy}{dt}\right] + [cy] = 0.$$

Notice that all the powers of x cancel and we end up with

$$a\frac{d^2y}{dt^2} + (b-a)\frac{dy}{dt} + cy = 0.$$

To solve this constant coefficient equation we look for solutions in the form $y = e^{rt}$ and we get characteristic equation $ar^2 + (b-a)r + c = 0$. The general solution is therefore determined by the discriminant *Discriminant*: $\Delta = (b-a)^2 - 4ac$. From College Algebra you may recall there are *Three Cases* depending on the sign of the discriminant:

A. $\Delta > 0$ Real distinct roots $r_1 \neq r_2 \ \Rightarrow$ (general solution) $y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$

B. $\Delta = 0$ Real double root $r_0 = r_1 = r_2 \ \Rightarrow$ (general solution) $y = c_1 e^{r_0 t} + c_2 x e^{r_0 t}$

C. $\Delta < 0$ Complet roots $r = \alpha \pm i\beta \ \Rightarrow$ (general solution) $y = c_1 e^{\alpha t}\cos(\beta t) + c_2 e^{\alpha t}\sin(\beta t)$

But we do not want the answers in terms of t so we must convert these formulas back to x using $x = e^t$ (and $t = \ln(x)$). Doing so gives exactly the formulas above in 1., 2. and 3. In particular

$$y = c_1 e^{r_1 t} + c_2 e^{r_2 t} = c_1 x^{r_1} + c_2 x^{r_2},$$

$$y = c_1 e^{r_0 t} + c_2 x e^{r_0 t} = c_1 x^{r_0} + c_2 \ln(x) x^{r_0}$$

and

$$y = c_1 e^{\alpha t}\cos(\beta t) + c_2 e^{\alpha t}\sin(\beta t) = c_1 x^{\alpha}\cos(\beta \ln(x)) + c_2 x^{\alpha}\sin(\beta \ln(x)).$$

Example 3.45. Consider $x^2y'' - 2y = 0$ which implies $a = 1$, $b = 0$ and $c = -2$ so the characteristic polynomial is $r^2 - r - 2 = 0$ which has roots $r = -1, 2$ so the general solution is

$$y = c_1 x^2 + c_2 x^{-1}.$$

Now solve the IVP with $y(1) = 6$ and $y'(1) = 3$: We have $y' = 2c_1 x - c_2 x^{-2}$ so

$$c_1 + c_2 = 6$$
$$2c_1 - c_2 = 3$$

so that $3c_1 = 9 \;\Rightarrow\; c_1 = 3 \;\Rightarrow c_2 = 3$ and we have $y = 3x^2 + 3x^{-1}$.

Example 3.46. Consider $x^2y'' + xy' + 4y = 0$ which implies $a = 1$, $b = 1$ and $c = 4$ so the characteristic polynomial is $r^2 + 4 = 0$ which has roots $r = 0 \pm 2i$ so the general solution is

$$y = c_1 \cos(2\ln(x)) + c_2 \sin(2\ln(x)).$$

Example 3.47. Consider $x^2y'' - 3xy' + 4y = 0$ which implies $a = 1$, $b = -3$ and $c = 4$ so the characteristic polynomial is $r^2 - 4r + 4 = 0$ which has a double root $r = 2, 2$ so the general solution is

$$y = c_1 x^2 + c_2 \ln(x)\, x^2.$$

Example 3.48. Consider $x^2y'' + 3xy' + 2y = 0$ which implies $a = 1$, $b = 3$ and $c = 2$ so the characteristic polynomial is $r^2 + 2r + 2 = 0$ which can be written as

$$r^2 - 2(-1)r + (-1)^2 + (1)^2 = 0$$

so it has complex roots with $\alpha = -1$ and $\beta = 1$ so that $r = -1 \pm i$ and the general solution is

$$y = c_1 x^{-x}\cos(\ln(x)) + c_2 x^{-x}\sin(\ln(x)).$$

Example 3.49. Suppose we are give $x^2y'' + xy' + y = \sec(\ln(x))$. First we consider the homogeneous problem $x^2y'' + xy' + y = 0$ so that the auxiliary equation is $r^2 + 1 = 0$

so that $r = 0 \pm i$. In this case we can take $y_1 = \cos(\ln(x))$ and $y_2 = \sin(\ln(x))$ so the complementary solution is $y_c = c_1 \cos(\ln(x)) + c_2 \sin(\ln(x))$. Next for variation of parameters we need to write the equation in the correct form by dividing by x^2 to obtain

$$y'' + \frac{1}{x}y' + \frac{1}{x^2}y = \frac{\sec(\ln(x))}{x^2}.$$

In this way we see that $f(x) = \sec(\ln(x))/x^2$. Next we compute the Wronskian

$$W(x) = \begin{vmatrix} \cos(\ln(x)) & \sin(\ln(x)) \\ -\sin(\ln(x))/x & \cos(\ln(x))/x \end{vmatrix} = \frac{\cos^2(\ln(x)) + \sin^2(\ln(x))}{x} = \frac{1}{x}.$$

$$u' = \frac{-\sin(\ln(x))\sec(\ln(x)/x^2)}{1/x} = \frac{-\sin(\ln(x))}{x\cos(\ln(x))}, \quad \Rightarrow \quad u = -\int \frac{\sin(\ln(x))}{x\cos(\ln(x))}\,dx = \ln(\cos(\ln(x))).$$

$$v' = \frac{\cos(\ln(x))\sec(\ln(x)/x^2)}{1/x} = 1/x, \quad \Rightarrow \quad v = \int \frac{1}{x}\,dx = \ln(x).$$

So we have

$$y_p = \cos(\ln(x))\ \ln(\cos(\ln(x))) + \ln(x)\sin(\ln(x)).$$

Example 3.50. Suppose we are give $x^2y'' - xy' + y = 2x$. First we consider the homogeneous problem $x^2y'' - xy' + y = 0$ so that the auxiliary equation is $r^2 - 2r + 1 = 0$ so that $r = 1, 1$. In this case we can take $y_1 = x$ and $y_2 = x\ln(x)$ so the complementary solution is $y_c = c_1x + c_2x\ln(x)$. Next for variation of parameters we need to write the equation in the correct form by dividing by x^2 to obtain

$$y'' - \frac{1}{x}y' + \frac{1}{x^2}y = \frac{2}{x}.$$

In this way we see that $f(x) = 2/x$. Next we compute the Wronskian

$$W(x) = \begin{vmatrix} x & x\ln(x) \\ 1 & 1 + \ln(x) \end{vmatrix} = x.$$

$$y_p = -x\int \frac{x\ln(x)(2/x)}{x}\,dx + x\ln(x)\int \frac{x(2/x)}{x}\,dx$$
$$= -2x\int \frac{\ln(x)}{x}\,dx + x\ln(x)2\int \frac{dx}{x}$$

(in the first intrgral set $u = \ln(x) \;\Rightarrow\; du = dx/x$)

$$-2\int u\,du + 2x(\ln(x))^2 = -u^2 + 2x(\ln(x))^2 = -x(\ln(x))^2 + 2x(\ln(x))^2$$
$$= x(\ln(x))^2$$

Example 3.51. Suppose we are give $x^2y'' - 3xy' + 3y = 2x^4e^x$ with $y(1) = -4$ and $y'(1) = 2e^1$. First we consider the homogeneous problem $x^2y'' - 3xy' + 3y = 0$ so that the auxiliary equation is $r^2 - 4r + 3 = 0$ so that $r = 1, 3$. In this case we can take $y_1 = x$ and $y_2 = x^3$ so the complementary solution is $y_c = c_1x + c_2x^3$. Next for variation of parameters we need to write the equation in the correct form by dividing by x^2 to obtain

$$y'' - \frac{3}{x}y' + \frac{3}{x^2}y = 2x^2e^x.$$

In this way we see that $f(x) = 2x^2e^x$. Next we compute the Wronskian

$$W(x) = \begin{vmatrix} x & x^3 \\ 1 & 3x^2 \end{vmatrix} = 2x^3.$$

$$y_p = -x\int \frac{x^3(2x^2e^x)}{2x^3}\,dx + x^3\int \frac{x(2x^2e^x)}{2x^3}\,dx$$
$$= -x\int x^2e^x\,dx + x^3\int e^x\,dx$$
$$= -x(x^2 - 2x + 2)e^x + x^3e^x = (2x^2 - 2x)e^x,$$

where above we have applied integration by parts twice to compute $\int x^2e^x\,dx$:

$$\int x^2e^x\,dx = \int x^2(e^x)'\,dx = x^2e^x - \int 2xe^x\,dx$$
$$= x^2e^x - 2\int x(e^x)'\,dx = x^2e^x - 2\left[xe^x - \int e^x\,dx\right] = (x^2 - 2x + 2)e^x$$

Therefore the general solution is

$$y = c_1 x + c_2 x^3 + (2x^2 - 2x)e^x$$

and so

$$y' = c_1 + 3c_2 x^2 + (2x^2 + 2x - 2)e^x.$$

Applying the initial conditions we have

$$\begin{aligned} c_1 + c_2 + 2 &= -4 \\ c_1 + 3c_2 + 2e^1 &= 2e^1 \end{aligned} \quad \text{or} \quad \begin{aligned} c_1 + c_2 &= -6 \\ c_1 + 3c_2 &= 0 \end{aligned}.$$

Multiplying the second equation by -1 and adding to the first equation we have

$$-2c_2 = -4 \quad \Rightarrow \quad c_2 = 2. \quad \text{so then } c_1 = -6.$$

Therefore the unique solution of the IVP is $y = -6x + 2x^3 + (2x^2 - 2x)e^x$.