SUMMATION IDENTITIES AND SPECIAL VALUES OF HYPERGEOMETRIC SERIES IN THE p-ADIC SETTING

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Abstract. We prove hypergeometric type summation identities for a function defined in terms of quotients of the p-adic gamma function by counting points on certain families of hyperelliptic curves over $\mathbb{F}_q$. We also find certain special values of that function.

1. Introduction and statement of results

In [11], Greene introduced the notion of hypergeometric functions over finite fields analogous to classical hypergeometric series. Since then many interesting relations between special values of these hypergeometric functions and the number of points on certain varieties over finite fields have been obtained. The arguments of these functions are multiplicative characters of finite fields and, consequently, results involving hypergeometric functions over finite fields are often restricted to primes in certain congruence classes. For example, the expressions for the trace of Frobenius map on families of elliptic curves given in [1, 2, 9, 15, 16] are restricted to certain congruence classes to facilitate the existence of characters of specific orders. To overcome these restrictions, in [18, 19], the third author defined a function ${}_nG_n[\cdots]$ in terms of quotients of the p-adic gamma function which can best be described as an analogue of hypergeometric series in the p-adic setting. He showed [17, 18, 19] how results involving hypergeometric functions over finite fields can be extended to almost all primes using the function ${}_nG_n[\cdots]$.

Let p be an odd prime, and let $\mathbb{F}_q$ denote the finite field with q elements. Let $\Gamma_p(\cdot)$ denote the Morita's p-adic gamma function, and let ω denote the Teichmüller character of $\mathbb{F}_q$ with $\overline{\omega}$ denoting its character inverse. For $x \in \mathbb{Q}$ we let $\lfloor x \rfloor$ denote the greatest integer less than or equal to x and $\langle x \rangle$ denote the fractional part of x.

Definition 1.1. [19, Definition 5.1] Let $q = p^r$, for p an odd prime and $r \in \mathbb{Z}^+$, and let $t \in \mathbb{F}_q$. For $n \in \mathbb{Z}^+$ and $1 \leq i \leq n$, let $a_i, b_i \in \mathbb{Q} \cap \mathbb{Z}_p$. Then we define

$$
{}_nG_n\left[\begin{matrix} a_1, & a_2, & \dots, & a_n \\ b_1, & b_2, & \dots, & b_n \end{matrix} \,|t\right]_q := \frac{-1}{q-1}\sum_{j=0}^{q-2}(-1)^{jn}\,\overline{\omega}^j(t)
$$
$$
\times \prod_{i=1}^{n}\prod_{k=0}^{r-1}(-p)^{-\lfloor\langle a_ip^k\rangle - \frac{jp^k}{q-1}\rfloor - \lfloor\langle -b_ip^k\rangle + \frac{jp^k}{q-1}\rfloor}\,\frac{\Gamma_p(\langle(a_i - \frac{j}{q-1})p^k\rangle)}{\Gamma_p(\langle a_ip^k\rangle)}\,\frac{\Gamma_p(\langle(-b_i + \frac{j}{q-1})p^k\rangle)}{\Gamma_p(\langle -b_ip^k\rangle)}.
$$

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We note that the value of ${}_nG_n[\cdots]$ depends only on the fractional part of the parameters a_i and b_i, and is invariant if we change the order of the parameters.

The aim of this paper is to prove summation identities for ${}_nG_n[\cdots]$. In [19], the third author showed that transformations for hypergeometric functions over finite fields can be re-written in terms of ${}_nG_n[\cdots]$. However, such transformations will only hold for all p where the original characters existed over $\mathbb{F}_q$, and hence restricted to primes in certain congruence classes. It is a non-trivial exercise to then extend these results to almost all primes. While numerous transformations exist for the finite field hypergeometric functions, very few exist for ${}_nG_n[\cdots]$ in full generality. The first and second authors [5, 6] provide transformations for ${}_2G_2[\cdots]_q$ by counting points on various families of elliptic curves over $\mathbb{F}_q$. Recently, the third author and Fuselier [10] provide two transformations for ${}_nG_n[\cdots]_p$ when $n = 3$ and $n = 4$, respectively. They also provide two transformations for ${}_nG_n[\cdots]_p$ for any n. In this paper we prove eight summation identities for the function ${}_nG_n[\cdots]_q$ for any n, which are listed below. Let ϕ be the quadratic character on $\mathbb{F}_q^\times$ extended to all of $\mathbb{F}_q$ by setting $\phi(0) := 0$.

Theorem 1.2. *Let $d \geq 4$ be even, and let p be an odd prime such that $p \nmid d(d-1)$. Let $a, b \in \mathbb{F}_q^\times$. For $y \in \mathbb{F}_q$, let $f(y) = \frac{d}{a}\left(\frac{(b-y^2)d}{a(d-1)}\right)^{d-1}$ and $g(y) = \frac{d(b-y^2)}{a}\left(\frac{d}{a(d-1)}\right)^{d-1}$. Let $l = \gcd(d-1, q-1)$, and let χ be a multiplicative character of order l. If b is not a square in $\mathbb{F}_q$ then*

$$\sum_{y\in\mathbb{F}_q}\phi(y^2-b)$$
$$\times\ {}_{d-1}G_{d-1}\left[\begin{array}{ccccccc} \frac{1}{2(d-1)}, & \frac{3}{2(d-1)}, & \ldots, & \frac{d-1}{2(d-1)}, & \frac{d+1}{2(d-1)}, & \ldots, & \frac{2d-3}{2(d-1)} \\ 0, & \frac{1}{d}, & \ldots, & \frac{d-2}{2d}, & \frac{d+2}{2d}, & \ldots, & \frac{d-1}{d} \end{array} \Big| f(y)\right]_q$$
$$= -1 - q\cdot {}_{d-2}G_{d-2}\left[\begin{array}{ccccccc} \frac{1}{d-1}, & \frac{2}{d-1}, & \ldots, & \frac{d-2}{2(d-1)}, & \frac{d}{2(d-1)}, & \ldots, & \frac{d-2}{d-1} \\ \frac{1}{d}, & \frac{2}{d}, & \ldots, & \frac{d-2}{2d}, & \frac{d+2}{2d}, & \ldots, & \frac{d-1}{d} \end{array} \Big| f(0)\right]_q$$

and

$$\sum_{y\in\mathbb{F}_q}\phi(y^2-b)$$
$$\times\ {}_{d-1}G_{d-1}\left[\begin{array}{ccccccc} \frac{1}{2(d-1)}, & \frac{3}{2(d-1)}, & \ldots, & \frac{d-1}{2(d-1)}, & \frac{d+1}{2(d-1)}, & \ldots, & \frac{2d-3}{2(d-1)} \\ 0, & \frac{1}{d}, & \ldots, & \frac{d-2}{2d}, & \frac{d+2}{2d}, & \ldots, & \frac{d-1}{d} \end{array} \Big| g(y)\right]_q$$
$$= -1 + q\cdot {}_{d-2}G_{d-2}\left[\begin{array}{ccccccc} \frac{1}{(d-1)}, & \frac{2}{(d-1)}, & \ldots, & \frac{d-2}{2(d-1)}, & \frac{d}{2(d-1)}, & \ldots, & \frac{d-2}{d-1} \\ \frac{1}{d}, & \frac{2}{d}, & \ldots, & \frac{d-2}{2d}, & \frac{d+2}{2d}, & \ldots, & \frac{d-1}{d} \end{array} \Big| g(0)\right]_q.$$

If b is a square in $\mathbb{F}_q$ then

$$1 + 2\sum_{j=0}^{l-1}\chi^j(-a) + \sum_{\substack{y\in\mathbb{F}_q \\ y\neq\pm\sqrt{b}}}\phi(y^2-b)$$
$$\times\ {}_{d-1}G_{d-1}\left[\begin{array}{ccccccc} \frac{1}{2(d-1)}, & \frac{3}{2(d-1)}, & \ldots, & \frac{d-1}{2(d-1)}, & \frac{d+1}{2(d-1)}, & \ldots, & \frac{2d-3}{2(d-1)} \\ 0, & \frac{1}{d}, & \ldots, & \frac{d-2}{2d}, & \frac{d+2}{2d}, & \ldots, & \frac{d-1}{d} \end{array} \Big| f(y)\right]_q$$

$$= -q \cdot {}_{d-2}G_{d-2}\left[\begin{matrix} \frac{1}{d-1}, & \frac{2}{d-1}, & \dots, & \frac{d-2}{2(d-1)}, & \frac{d}{2(d-1)}, & \dots, & \frac{d-2}{d-1} \\ \frac{1}{d}, & \frac{2}{d}, & \dots, & \frac{d-2}{2d}, & \frac{d+2}{2d}, & \dots, & \frac{d-1}{d} \end{matrix} \;|f(0)\right]_q$$

and

$$\begin{aligned} &3 + \sum_{\substack{y\in\mathbb{F}_q \\ y\neq\pm\sqrt{b}}} \phi(y^2-b) \\ &\times {}_{d-1}G_{d-1}\left[\begin{matrix} \frac{1}{2(d-1)}, & \frac{3}{2(d-1)}, & \dots, & \frac{d-1}{2(d-1)}, & \frac{d+1}{2(d-1)}, & \dots, & \frac{2d-3}{2(d-1)} \\ 0, & \frac{1}{d}, & \dots, & \frac{d-2}{2d}, & \frac{d+2}{2d}, & \dots, & \frac{d-1}{d} \end{matrix} \;|g(y)\right]_q \\ &= -q \cdot {}_{d-2}G_{d-2}\left[\begin{matrix} \frac{1}{(d-1)}, & \frac{2}{(d-1)}, & \dots, & \frac{d-2}{2(d-1)}, & \frac{d}{2(d-1)}, & \dots, & \frac{d-2}{(d-1)} \\ \frac{1}{d}, & \frac{2}{d}, & \dots, & \frac{d-2}{2d}, & \frac{d+2}{2d}, & \dots, & \frac{d-1}{d} \end{matrix} \;|g(0)\right]_q. \end{aligned}$$

Theorem 1.3. *Let $d \geq 3$ be odd, and let p be an odd prime such that $p \nmid d(d-1)$. Let $a, b \in \mathbb{F}_q^{\times}$. For $y \in \mathbb{F}_q$, let $f(y) = \frac{d}{a}\left(\frac{(b-y^2)d}{a(d-1)}\right)^{d-1}$ and $g(y) = \frac{d(b-y^2)}{a}\left(\frac{d}{a(d-1)}\right)^{d-1}$. Let $l = \gcd(d-1, q-1)$, and let χ be a multiplicative character of order l.*
If b is not a square in $\mathbb{F}_q$ then

$$\begin{aligned} &\sum_{y\in\mathbb{F}_q} {}_{d-1}G_{d-1}\left[\begin{matrix} 0, & \frac{1}{d-1}, & \dots, & \frac{d-3}{2(d-1)}, & \frac{d-1}{2(d-1)}, & \dots, & \frac{d-2}{d-1} \\ \frac{1}{2d}, & \frac{3}{2d}, & \dots, & \frac{d-2}{2d}, & \frac{d+2}{2d}, & \dots, & \frac{2d-1}{2d} \end{matrix} \;|-f(y)\right]_q \\ &= q \cdot {}_{d-1}G_{d-1}\left[\begin{matrix} \frac{1}{2(d-1)}, & \frac{3}{2(d-1)}, & \dots, & \frac{d-2}{2(d-1)}, & \frac{d}{2(d-1)}, & \dots, & \frac{2d-3}{2(d-1)} \\ \frac{1}{2d}, & \frac{3}{2d}, & \dots, & \frac{d-2}{2d}, & \frac{d+2}{2d}, & \dots, & \frac{2d-1}{2d} \end{matrix} \;|-f(0)\right]_q \end{aligned}$$

and

$$\begin{aligned} &\frac{\phi(a)}{q} \cdot \sum_{y\in\mathbb{F}_q} \phi(y^2-b) \\ &\times {}_{d-1}G_{d-1}\left[\begin{matrix} 0, & \frac{1}{d-1}, & \dots, & \frac{d-3}{2(d-1)}, & \frac{d-1}{2(d-1)}, & \dots, & \frac{d-2}{d-1} \\ \frac{1}{2d}, & \frac{3}{2d}, & \dots, & \frac{d-2}{2d}, & \frac{d+2}{2d}, & \dots, & \frac{2d-1}{2d} \end{matrix} \;|-g(y)\right]_q \\ &= {}_{d-1}G_{d-1}\left[\begin{matrix} \frac{1}{d-1}, & \frac{2}{d-1}, & \dots, & \frac{d-1}{2(d-1)}, & \frac{d+1}{2(d-1)}, & \dots, & \frac{d-2}{d-1}, & \frac{1}{2} \\ \frac{1}{d}, & \frac{2}{d}, & \dots, & \frac{d-1}{2d}, & \frac{d+1}{2d}, & \dots, & \frac{d-2}{d}, & \frac{d-1}{d} \end{matrix} \;|-g(0)\right]_q. \end{aligned}$$

If b is a square in $\mathbb{F}_q$ then

$$\begin{aligned} &\sum_{\substack{y\in\mathbb{F}_q \\ y\neq\pm\sqrt{b}}} {}_{d-1}G_{d-1}\left[\begin{matrix} 0, & \frac{1}{d-1}, & \dots, & \frac{d-3}{2(d-1)}, & \frac{d-1}{2(d-1)}, & \dots, & \frac{d-2}{d-1} \\ \frac{1}{2d}, & \frac{3}{2d}, & \dots, & \frac{d-2}{2d}, & \frac{d+2}{2d}, & \dots, & \frac{2d-1}{2d} \end{matrix} \;|-f(y)\right]_q \\ &= -2\sum_{j=0}^{l-1}(\chi^j\phi)(-a) - q \\ &\times {}_{d-1}G_{d-1}\left[\begin{matrix} \frac{1}{2(d-1)}, & \frac{3}{2(d-1)}, & \dots, & \frac{d-2}{2(d-1)}, & \frac{d}{2(d-1)}, & \dots, & \frac{2d-3}{2(d-1)} \\ \frac{1}{2d}, & \frac{3}{2d}, & \dots, & \frac{d-2}{2d}, & \frac{d+2}{2d}, & \dots, & \frac{2d-1}{2d} \end{matrix} \;|-f(0)\right]_q \end{aligned}$$

and

$$-\frac{2}{q}-\frac{\phi(a)}{q}\sum_{\substack{y\in\mathbb{F}_q\\ y\neq\pm\sqrt{b}}}\phi(y^2-b)$$
$$\times\, {}_{d-1}G_{d-1}\left[\begin{matrix} 0, & \frac{1}{d-1}, & \ldots, & \frac{d-3}{2(d-1)}, & \frac{d-1}{2(d-1)}, & \ldots, & \frac{d-2}{d-1} \\ \frac{1}{2d}, & \frac{3}{2d}, & \ldots, & \frac{d-2}{2d}, & \frac{d+2}{2d}, & \ldots, & \frac{2d-1}{2d}\end{matrix} \mid -g(y)\right]_q$$
$$= {}_{d-1}G_{d-1}\left[\begin{matrix} \frac{1}{d-1}, & \frac{2}{d-1}, & \ldots, & \frac{d-1}{2(d-1)}, & \frac{d+1}{2(d-1)}, & \ldots, & \frac{d-2}{d-1}, & \frac{1}{2} \\ \frac{1}{d}, & \frac{2}{d}, & \ldots, & \frac{d-1}{2d}, & \frac{d+1}{2d}, & \ldots, & \frac{d-2}{d}, & \frac{d-1}{d}\end{matrix} \mid -g(0)\right]_q.$$

We now give one example to show how the above theorems are applied in specific cases.

Example 1.4. Let $a\neq 0$ and $p\geq 5$. Taking $d=3$ and $b=1$ in Theorem 1.3, we deduce that

$$\sum_{\substack{y\in\mathbb{F}_q\\ y\neq\pm 1}} {}_2G_2\left[\begin{matrix} 0, & \frac{1}{2} \\ \frac{1}{6}, & \frac{5}{6}\end{matrix} \mid -\frac{27}{4a^3}(1-y^2)^2\right]_q = -2-2\phi(-a)-q\cdot {}_2G_2\left[\begin{matrix} \frac{1}{4}, & \frac{3}{4} \\ \frac{1}{6}, & \frac{5}{6}\end{matrix} \mid -\frac{27}{4a^3}\right]_q$$

and

$$\sum_{\substack{y\in\mathbb{F}_q\\ y\neq\pm 1}} \phi(y^2-1)\, {}_2G_2\left[\begin{matrix} 0, & \frac{1}{2} \\ \frac{1}{6}, & \frac{5}{6}\end{matrix} \mid -\frac{27}{4a^3}(1-y^2)\right]_q = -2\phi(a)-q\phi(a)\cdot {}_2G_2\left[\begin{matrix} \frac{1}{2}, & \frac{1}{2} \\ \frac{1}{3}, & \frac{2}{3}\end{matrix} \mid -\frac{27}{4a^3}\right]_q.$$

We also derive the following transformation for ${}_2G_2[\cdots]$.

Theorem 1.5. *Let $q=p^r$, $p>3$ be a prime. Let $a,b\in\mathbb{F}_q^\times$ and $\frac{-27b^2}{4a^3}\neq 1$. Then*

$${}_2G_2\left[\begin{matrix} \frac{1}{4}, & \frac{3}{4} \\ \frac{1}{3}, & \frac{2}{3}\end{matrix} \mid \frac{-27b^2}{4a^3}\right]_q = \phi(-a)\cdot {}_2G_2\left[\begin{matrix} \frac{1}{4}, & \frac{3}{4} \\ \frac{1}{6}, & \frac{5}{6}\end{matrix} \mid \frac{-27b^2}{4a^3}\right]_q.$$

In particular, if $\frac{27b^2}{4a^6}\neq 1$ then

$${}_2G_2\left[\begin{matrix} \frac{1}{4}, & \frac{3}{4} \\ \frac{1}{3}, & \frac{2}{3}\end{matrix} \mid \frac{27b^2}{4a^6}\right]_q = {}_2G_2\left[\begin{matrix} \frac{1}{4}, & \frac{3}{4} \\ \frac{1}{6}, & \frac{5}{6}\end{matrix} \mid \frac{27b^2}{4a^6}\right]_q.$$

Remark 1.6. In [5, 6], the first and second authors derived transformations for ${}_2G_2[\cdots]_q$ with different parameters. Thus, we can derive many such transformations by combining the transformation of Theorem 1.5 with those in [5, 6].

Let $d\geq 2$. In [4], the first and second authors expressed the number of distinct zeros of the polynomials x^d+ax+b and $x^d+ax^{d-1}+b$ over $\mathbb{F}_q$ in terms ${}_{d-1}G_{d-1}[\cdots]$. We now state four theorems from [4] which we will need to prove our main results.

Theorem 1.7. ([4, Theorem 1.2]) *Let $d\geq 2$ be even, and let p be an odd prime such that $p\nmid d(d-1)$. Let $a,b\in\mathbb{F}_q^\times$. If $N(x^d+ax+b=0)$ denotes the number of distinct zeros of the polynomial x^d+ax+b in $\mathbb{F}_q$ then*

$$N(x^d+ax+b=0)=1+\phi(-b)$$
$$\times\, {}_{d-1}G_{d-1}\left[\begin{matrix} \frac{1}{2(d-1)}, & \frac{3}{2(d-1)}, & \ldots, & \frac{d-1}{2(d-1)}, & \frac{d+1}{2(d-1)}, & \ldots, & \frac{2d-3}{2(d-1)} \\ 0, & \frac{1}{d}, & \ldots, & \frac{\frac{d}{2}-1}{d}, & \frac{\frac{d}{2}+1}{d}, & \ldots, & \frac{d-1}{d}\end{matrix} \mid \alpha\right]_q,$$

where $\alpha = \frac{d}{a}\left(\frac{bd}{a(d-1)}\right)^{d-1}$.

Theorem 1.8. ([4, Theorem 1.3]) *Let $d \geq 3$ be odd, and let p be an odd prime such that $p \nmid d(d-1)$. Let $a, b \in \mathbb{F}_q^{\times}$. If $N(x^d + ax + b = 0)$ denotes the number of distinct zeros of the polynomial $x^d + ax + b$ in $\mathbb{F}_q$ then*

$$N(x^d + ax + b = 0) = 1 + \phi(-a)$$
$$\times {}_{d-1}G_{d-1}\left[\begin{array}{ccccccc} 0, & \frac{1}{d-1}, & \ldots, & \frac{(d-3)/2}{d-1}, & \frac{(d-1)/2}{d-1}, & \ldots, & \frac{d-2}{d-1} \\ \frac{1}{2d}, & \frac{3}{2d}, & \ldots, & \frac{d-2}{2d}, & \frac{d+2}{2d}, & \ldots, & \frac{2d-1}{2d} \end{array} \mid -\alpha\right]_q,$$

where $\alpha = \frac{d}{a}\left(\frac{bd}{a(d-1)}\right)^{d-1}$.

Theorem 1.9. ([4, Theorem 1.4]) *Let $d \geq 2$ be even, and let p be an odd prime such that $p \nmid d(d-1)$. Let $a, b \in \mathbb{F}_q^{\times}$. If $N(x^d + ax^{d-1} + b = 0)$ denotes the number of distinct zeros of the polynomial $x^d + ax^{d-1} + b$ in $\mathbb{F}_q$ then*

$$N(x^d + ax^{d-1} + b = 0) = 1 + \phi(-b)$$
$$\times {}_{d-1}G_{d-1}\left[\begin{array}{ccccccc} \frac{1}{2(d-1)}, & \frac{3}{2(d-1)}, & \ldots, & \frac{d-1}{2(d-1)}, & \frac{d+1}{2(d-1)}, & \ldots, & \frac{2d-3}{2(d-1)} \\ 0, & \frac{1}{d}, & \ldots, & \frac{\frac{d}{2}-1}{d}, & \frac{\frac{d}{2}+1}{d}, & \ldots, & \frac{d-1}{d} \end{array} \mid \beta\right]_q,$$

where $\beta = \frac{bd}{a}\left(\frac{d}{a(d-1)}\right)^{d-1}$.

Theorem 1.10. ([4, Theorem 1.5]) *Let $d \geq 3$ be odd, and let p be an odd prime such that $p \nmid d(d-1)$. Let $a, b \in \mathbb{F}_q^{\times}$. If $N(x^d + ax^{d-1} + b = 0)$ denotes the number of distinct zeros of the polynomial $x^d + ax^{d-1} + b$ in $\mathbb{F}_q$ then*

$$N(x^d + ax^{d-1} + b = 0) = 1 + \phi(-ab)$$
$$\times {}_{d-1}G_{d-1}\left[\begin{array}{ccccccc} 0, & \frac{1}{d-1}, & \ldots, & \frac{(d-3)/2}{d-1}, & \frac{(d-1)/2}{d-1}, & \ldots, & \frac{d-2}{d-1} \\ \frac{1}{2d}, & \frac{3}{2d}, & \ldots, & \frac{d-2}{2d}, & \frac{d+2}{2d}, & \ldots, & \frac{2d-1}{2d} \end{array} \mid -\beta\right]_q,$$

where $\beta = \frac{bd}{a}\left(\frac{d}{a(d-1)}\right)^{d-1}$.

2. Notations and Preliminaries

Throughout this paper p will denote an odd prime, $\mathbb{F}_q$ the finite field of $q = p^r$ elements, $\mathbb{Z}_p$ the ring of p-adic integers, $\mathbb{Q}_p$ the field of p-adic numbers, $\overline{\mathbb{Q}_p}$ the algebraic closure of $\mathbb{Q}_p$, and $\mathbb{C}_p$ the completion of $\overline{\mathbb{Q}_p}$. Let $\mathbb{Z}_q$ be the ring of integers in the unique unramified extension of $\mathbb{Q}_p$ with residue field $\mathbb{F}_q$. Let μ_{q-1} be the group of $(q-1)$-th roots of unity in $\mathbb{C}^{\times}$.

2.1. **Multiplicative characters and Gauss sums.** Let $\widehat{\mathbb{F}_q^{\times}}$ denote the group of multiplicative characters χ on $\mathbb{F}_q^{\times}$ with values in μ_{q-1}. $\widehat{\mathbb{F}_q^{\times}}$ is a cyclic group of order $q-1$. Let ε and ϕ denote the trivial and quadratic characters, respectively. The domain of each $\chi \in \widehat{\mathbb{F}_q^{\times}}$ is extended to $\mathbb{F}_q$ by setting $\chi(0) := 0$. We now state the *orthogonality relations* for multiplicative characters in the following lemma.

Lemma 2.1. ([13, Chapter 8]). *We have*

(1) $\displaystyle\sum_{x\in\mathbb{F}_q}\chi(x)=\begin{cases} q-1 & \text{if } \chi=\varepsilon;\\ 0 & \text{if } \chi\neq\varepsilon.\end{cases}$

(2) $\displaystyle\sum_{\chi\in\widehat{\mathbb{F}_q^{\times}}}\chi(x)=\begin{cases} q-1 & \text{if } x=1;\\ 0 & \text{if } x\neq 1.\end{cases}$

It is known that $\mathbb{Z}_q^{\times}$ contains all the $(q-1)$-th roots of unity. Therefore, we can consider multiplicative characters on $\mathbb{F}_q^{\times}$ to be maps $\chi:\mathbb{F}_q^{\times}\to\mathbb{Z}_q^{\times}$.

We now introduce some properties of Gauss sums. For further details, see [7] noting that we have adjusted results to take into account $\varepsilon(0)=0$. Let ζ_p be a fixed primitive p-th root of unity in $\overline{\mathbb{Q}_p}$. The trace map $\mathrm{tr}:\mathbb{F}_q\to\mathbb{F}_p$ is given by

$$\mathrm{tr}(\alpha)=\alpha+\alpha^p+\alpha^{p^2}+\cdots+\alpha^{p^{r-1}}.$$

Then the additive character $\theta:\mathbb{F}_q\to\mathbb{Q}_p(\zeta_p)$ is defined by $\theta(\alpha):=\zeta_p^{\mathrm{tr}(\alpha)}$. It is easy to see that $\theta(a+b)=\theta(a)\theta(b)$ and

$$\sum_{x\in\mathbb{F}_q}\theta(x)=0. \tag{2.1}$$

For $\chi\in\widehat{\mathbb{F}_q^{\times}}$, the *Gauss sum* is defined by

$$G(\chi):=\sum_{x\in\mathbb{F}_q}\chi(x)\theta(x).$$

If ζ_{q-1} is a primitive $(q-1)$-th root of unity in $\overline{\mathbb{Q}_p}$, then $G(\chi)$ lies in $\mathbb{Q}_p(\zeta_p,\zeta_{q-1})$. We let T denote a fixed generator of $\widehat{\mathbb{F}_q^{\times}}$ and denote by G_m the Gauss sum $G(T^m)$. Using (2.1) it is easy to show that $G_0=-1$. We will use the following results on Gauss sums in the proof of our main results.

Lemma 2.2. ([7, Theorem 1.1.4 (a)]). *If $k\in\mathbb{Z}$ and $T^k\neq\varepsilon$, then*

$$G_kG_{-k}=qT^k(-1).$$

Lemma 2.3. ([9, Lemma 2.2]). *For all $\alpha\in\mathbb{F}_q^{\times}$,*

$$\theta(\alpha)=\frac{1}{q-1}\sum_{m=0}^{q-2}G_{-m}T^m(\alpha).$$

Theorem 2.4. ([7, Davenport-Hasse Relation, Thm 11.3.5]). *Let k be a positive integer and let $q=p^r$ be a prime power such that $q\equiv 1 \pmod k$. For $\chi,\psi\in\widehat{\mathbb{F}_q^{\times}}$,*

$$\prod_{\chi^k=\varepsilon}G(\chi\psi)=-G(\psi^k)\psi(k^{-k})\prod_{\chi^k=\varepsilon}G(\chi).$$

2.2. **p-adic gamma function and Gross-Koblitz formula.** Let $\omega:\mathbb{F}_q^{\times}\to\mathbb{Z}_q^{\times}$ be the Teichmüller character. For $a\in\mathbb{F}_q^{\times}$, the value $\omega(a)$ is just the $(q-1)$-th root of unity in $\mathbb{Z}_q$ such that $\omega(a)\equiv a \pmod p$. We note that $\omega|_{\mathbb{F}_p^{\times}}$ is the Teichmüller character on $\mathbb{F}_p^{\times}$ with values in $\mathbb{Z}_p^{\times}$. Also, $\widehat{\mathbb{F}_q^{\times}}=\{\omega^j:0\leq j\leq q-2\}$.

We now recall the p-adic gamma function. For further details, see [14]. The p-adic gamma function Γ_p is defined by setting $\Gamma_p(0)=1$, and for $n\in\mathbb{Z}^+$ by

$$\Gamma_p(n):=(-1)^n\prod_{\substack{0<j<n\\ p\nmid j}}j.$$

If $x, y \in \mathbb{Z}^+$ and $x \equiv y \pmod{p^k\mathbb{Z}}$, then $\Gamma_p(x) \equiv \Gamma_p(y) \pmod{p^k\mathbb{Z}}$. Therefore, the function has a unique extension to a continuous function $\Gamma_p : \mathbb{Z}_p \to \mathbb{Z}_p^\times$. If $x \in \mathbb{Z}_p$ and $x \neq 0$, then $\Gamma_p(x)$ is defined as

$$\Gamma_p(x) := \lim_{x_n \to x} \Gamma_p(x_n),$$

where x_n runs through any sequence of positive integers p-adically approaching x. The following lemma will be used in the proofs of our main results.

Lemma 2.5. ([5, Lemma 3.1]). *Let p be a prime and $q = p^r$. For $0 \leq j \leq q-2$ and $t \in \mathbb{Z}^+$ with $p \nmid t$, we have*

$$\omega(t^{tj}) \prod_{i=0}^{r-1} \Gamma_p\left(\langle \frac{tp^i j}{q-1} \rangle\right) \prod_{h=1}^{t-1} \Gamma_p\left(\langle \frac{hp^i}{t} \rangle\right) = \prod_{i=0}^{r-1} \prod_{h=0}^{t-1} \Gamma_p\left(\langle \frac{p^i h}{t} + \frac{p^i j}{q-1} \rangle\right)$$

and

$$\omega(t^{-tj}) \prod_{i=0}^{r-1} \Gamma_p\left(\langle \frac{-tp^i j}{q-1} \rangle\right) \prod_{h=1}^{t-1} \Gamma_p\left(\langle \frac{hp^i}{t} \rangle\right) = \prod_{i=0}^{r-1} \prod_{h=0}^{t-1} \Gamma_p\left(\langle \frac{p^i(1+h)}{t} - \frac{p^i j}{q-1} \rangle\right).$$

The Gross-Koblitz formula, which is given below, allows us to relate Gauss sums and the p-adic gamma function. Let $\pi \in \mathbb{C}_p$ be the fixed root of $x^{p-1} + p = 0$ which satisfies $\pi \equiv \zeta_p - 1 \pmod{(\zeta_p - 1)^2}$. Recall that $\overline{\omega}$ denotes the character inverse of the Teichmüller character ω.

Theorem 2.6. ([12, Gross-Koblitz]). *For $a \in \mathbb{Z}$ and $q = p^r$,*

$$G(\overline{\omega}^a) = -\pi^{(p-1)\sum_{i=0}^{r-1} \langle \frac{ap^i}{q-1} \rangle} \prod_{i=0}^{r-1} \Gamma_p\left(\langle \frac{ap^i}{q-1} \rangle\right).$$

3. Proof of the results

Lemma 3.1. *Let p be an odd prime and $q = p^r$. Let $d \geq 4$ be even and $p \nmid d(d-1)$. For $1 \leq m \leq q-2$ such that $m \neq \frac{q-1}{2}$, $0 \leq i \leq r-1$ we have*

$$\begin{aligned} &\lfloor \frac{-2mp^i}{q-1} \rfloor + \lfloor \frac{mdp^i}{q-1} \rfloor + \lfloor \frac{-m(d-1)p^i}{q-1} \rfloor - \lfloor \frac{-mp^i}{q-1} \rfloor + 1 \\ &= \sum_{h=1}^{d-2} \lfloor \langle \frac{hp^i}{d-1} \rangle - \frac{mp^i}{q-1} \rfloor + \sum_{\substack{h=1 \\ h \neq \frac{d}{2}}}^{d-1} \lfloor \langle \frac{-hp^i}{d} \rangle + \frac{mp^i}{q-1} \rfloor. \end{aligned} \tag{3.1}$$

Proof. We express $\lfloor \frac{md(d-1)p^i}{q-1} \rfloor$ as $d(d-1)u + v$ for some $u, v \in \mathbb{Z}$, where $0 \leq v < d(d-1)$. By considering the cases $v = 0, 1, \ldots, d(d-1) - 1$ separately, we verify (3.1). For example, when $v = 0$ we deduce that $\lfloor \frac{-2mp^i}{q-1} \rfloor = -2u - 1$, $\lfloor \frac{mdp^i}{q-1} \rfloor = du$, $\lfloor \frac{-m(d-1)p^i}{q-1} \rfloor = -(d-1)u - 1$ and $\lfloor \frac{-mp^i}{q-1} \rfloor = -u - 1$. Substituting all these values we find that the left hand side of (3.1) is equal to zero. Since $p \nmid d$ we have

$$\sum_{\substack{h=1 \\ h \neq \frac{d}{2}}}^{d-1} \lfloor \langle \frac{-hp^i}{d} \rangle + \frac{mp^i}{q-1} \rfloor = \sum_{\substack{h=1 \\ h \neq \frac{d}{2}}}^{d-1} \lfloor \langle \frac{h}{d} \rangle + \frac{mp^i}{q-1} \rfloor = (d-2)u. \tag{3.2}$$

Also, $p \nmid (d-1)$ and hence

$$\sum_{h=1}^{d-2}\lfloor\langle\frac{hp^i}{d-1}\rangle-\frac{mp^i}{q-1}\rfloor=\sum_{h=1}^{d-2}\lfloor\langle\frac{h}{d-1}\rangle-\frac{mp^i}{q-1}\rfloor=-(d-2)u. \tag{3.3}$$

Substituting for (3.2) and (3.3), we see that the right hand side of (3.1) is also equal to zero, when $v=0$. Similarly we check (3.1) for the other values of v. This completes the proof of the lemma. □

Lemma 3.2. *Let p be an odd prime and $q=p^r$. Let $d\geq 3$ be odd and $p\nmid d(d-1)$. For $1\leq m\leq q-2$ and $0\leq i\leq r-1$ we have*

$$\begin{aligned}&\lfloor\frac{-2mp^i}{q-1}\rfloor+\lfloor\frac{mdp^i}{q-1}\rfloor+\lfloor\frac{-m(d-1)p^i}{q-1}\rfloor-\lfloor\frac{-mp^i}{q-1}\rfloor+1\\&=\sum_{h=1}^{d-2}\lfloor\langle\frac{hp^i}{d-1}\rangle-\frac{mp^i}{q-1}\rfloor+\lfloor\langle\frac{p^i}{2}\rangle-\frac{mp^i}{q-1}\rfloor+\sum_{h=1}^{d-1}\lfloor\langle\frac{-hp^i}{d}\rangle+\frac{mp^i}{q-1}\rfloor.\end{aligned}$$

Proof. The proof is similar to that of Lemma 3.1. □

Lemma 3.3. *Let p be an odd prime and $q=p^r$. Let $d\geq 3$ be odd and $p\nmid d(d-1)$. For $0\leq m\leq q-2$ such that $m\neq\frac{q-1}{2}$, $0\leq i\leq r-1$ we have*

$$\begin{aligned}&\lfloor\frac{-2mp^i}{q-1}\rfloor+\lfloor\frac{2mdp^i}{q-1}\rfloor+\lfloor\frac{-2m(d-1)p^i}{q-1}\rfloor-\lfloor\frac{-mp^i}{q-1}\rfloor-\lfloor\frac{mdp^i}{q-1}\rfloor-\lfloor\frac{-m(d-1)p^i}{q-1}\rfloor\\&=\sum_{\substack{h=1\\h\ odd}}^{2d-3}\lfloor\langle\frac{hp^i}{2(d-1)}\rangle-\frac{mp^i}{q-1}\rfloor+\sum_{\substack{h=1\\h\ odd\\h\neq d}}^{2d-1}\lfloor\langle\frac{-hp^i}{2d}\rangle+\frac{mp^i}{q-1}\rfloor.\end{aligned}$$

Proof. The proof is similar to that of Lemma 3.1. □

Lemma 3.4. *For $0<m\leq q-2$ we have*

$$\prod_{i=0}^{r-1}\Gamma_p(\langle(1-\frac{m}{q-1})p^i\rangle)\Gamma_p(\langle\frac{mp^i}{q-1}\rangle)=(-1)^r\overline{\omega}^m(-1). \tag{3.4}$$

For $0\leq m\leq q-2$ such that $m\neq\frac{q-1}{2}$ we have

$$\prod_{i=0}^{r-1}\frac{\Gamma_p(\langle(\frac12-\frac{m}{q-1})p^i\rangle)\Gamma_p(\langle(\frac12+\frac{m}{q-1})p^i\rangle)}{\Gamma_p(\langle\frac{p^i}{2}\rangle)\Gamma_p(\langle\frac{p^i}{2}\rangle)}=\overline{\omega}^m(-1). \tag{3.5}$$

Proof. Consider

$$\begin{aligned}I_m&=\prod_{i=0}^{r-1}\Gamma_p(\langle(1-\frac{m}{q-1})p^i\rangle)\Gamma_p(\langle\frac{mp^i}{q-1}\rangle)=\prod_{i=0}^{r-1}\Gamma_p(\langle\frac{-mp^i}{q-1}\rangle)\Gamma_p(\langle\frac{mp^i}{q-1}\rangle)\\&=\frac{\pi^{(p-1)\sum_{i=0}^{r-1}\langle\frac{-mp^i}{q-1}\rangle}\prod_{i=0}^{r-1}\Gamma_p\left(\langle\frac{-mp^i}{q-1}\rangle\right)\pi^{(p-1)\sum_{i=0}^{r-1}\langle\frac{mp^i}{q-1}\rangle}\prod_{i=0}^{r-1}\Gamma_p\left(\langle\frac{mp^i}{q-1}\rangle\right)}{\pi^{(p-1)\sum_{i=0}^{r-1}\{\langle\frac{-mp^i}{q-1}\rangle+\langle\frac{mp^i}{q-1}\rangle\}}}.\end{aligned}$$

Now, applying Gross-Koblitz formula (Theorem 2.6), Lemma 2.2 and the fact that $\langle\frac{-mp^i}{q-1}\rangle+\langle\frac{mp^i}{q-1}\rangle=1$, we obtain

$$I_m=\frac{G(\overline{\omega}^{-m})G(\overline{\omega}^{m})}{(-p)^r}=\frac{q\cdot\overline{\omega}^m(-1)}{(-1)^r\cdot q}=(-1)^r\overline{\omega}^m(-1).$$

This completes the proof of (3.4). If $m=0$ then clearly (3.5) is true. For $m\neq\frac{q-1}{2}$, and using Lemma 2.5, we have

$$J_m=\prod_{i=0}^{r-1}\frac{\Gamma_p(\langle(\frac{1}{2}-\frac{m}{q-1})p^i\rangle)\Gamma_p(\langle(\frac{1}{2}+\frac{m}{q-1})p^i\rangle)}{\Gamma_p(\langle\frac{p^i}{2}\rangle)\Gamma_p(\langle\frac{p^i}{2}\rangle)}=\prod_{i=0}^{r-1}\frac{\Gamma_p(\langle\frac{-2mp^i}{q-1}\rangle)\Gamma_p(\langle\frac{2mp^i}{q-1}\rangle)}{\Gamma_p(\langle(1-\frac{m}{q-1})p^i\rangle)\Gamma_p(\langle\frac{mp^i}{q-1}\rangle)}.$$

Using (3.4), Gross-Koblitz formula (Theorem 2.6), Lemma 2.2, and the fact that $\langle\frac{-2mp^i}{q-1}\rangle+\langle\frac{2mp^i}{q-1}\rangle=1$, we have

$$\begin{aligned}J_m&=\frac{\pi^{(p-1)\sum_{i=0}^{r-1}\langle\frac{-2mp^i}{q-1}\rangle}\prod_{i=0}^{r-1}\Gamma_p\left(\langle\frac{-2mp^i}{q-1}\rangle\right)\pi^{(p-1)\sum_{i=0}^{r-1}\langle\frac{2mp^i}{q-1}\rangle}\prod_{i=0}^{r-1}\Gamma_p\left(\langle\frac{2mp^i}{q-1}\rangle\right)}{(-1)^r\overline{\omega}^m(-1)\pi^{(p-1)\sum_{i=0}^{r-1}\{\langle\frac{-2mp^i}{q-1}\rangle+\langle\frac{2mp^i}{q-1}\rangle\}}}\\&=\frac{G(\overline{\omega}^{\,-2m})G(\overline{\omega}^{\,2m})}{q\overline{\omega}^m(-1)}=\frac{q\,\overline{\omega}^{2m}(-1)}{q\,\overline{\omega}^m(-1)}=\overline{\omega}^m(-1).\end{aligned}$$

□

Lemma 3.5. ([7, Lemma 10.4.1]) *Let $\gamma\in\mathbb{F}_q^{\times}$, and let k be a positive integer. Let χ be a character on $\mathbb{F}_q$ of order $d=\gcd(k,q-1)$. Then the number of solutions $x\in\mathbb{F}_q$ of $x^k=\gamma$ is*

$$N(x^k=\gamma)=\sum_{j=0}^{d-1}\chi^j(\gamma).$$

To prove Theorem 1.2 and Theorem 1.3, we will first express the number of points on certain families of hyperelliptic curves over $\mathbb{F}_q$ in terms of the G-function. For $d\geq 2$ and $a,b\neq 0$, we consider the hyperelliptic curves E_d and E_d' over $\mathbb{F}_q$ given by

$$E_d: y^2=x^d+ax+b$$

and

$$E_d': y^2=x^d+ax^{d-1}+b,$$

respectively. Let N_d and N_d' denote the number of $\mathbb{F}_q$-points on the curves E_d and E_d', respectively. We now give explicit expressions for N_d and N_d' in terms of the G-function in the following theorems.

Theorem 3.6. *Let $d\geq 4$ be even, and let p be an odd prime such that $p\nmid d(d-1)$. Then*

$$\begin{aligned}N_d&=q-1-q\\&\times{}_{d-2}G_{d-2}\left[\begin{matrix}\frac{1}{d-1}, & \frac{2}{d-1}, & \ldots, & \frac{d-2}{2(d-1)}, & \frac{d}{2(d-1)}, & \ldots, & \frac{d-2}{d-1}\\ \frac{1}{d}, & \frac{2}{d}, & \ldots, & \frac{d-2}{2d}, & \frac{d+2}{2d}, & \ldots, & \frac{d-1}{d}\end{matrix}\ |f(0)\right]_q,\end{aligned}$$

where f is defined as in Theorem 1.2.

Proof. Let $E_d(x,y)=x^d+ax+b-y^2$. Using the identity

$$\sum_{z\in\mathbb{F}_q}\theta(zE_d(x,y))=\begin{cases}q, & \text{if } E_d(x,y)=0;\\ 0, & \text{if } E_d(x,y)\neq 0,\end{cases}$$

we obtain

$$\begin{aligned} q\cdot N_d &= \sum_{x,y,z\in\mathbb{F}_q}\theta(zE_d(x,y)) \\ &= q^2+\sum_{z\in\mathbb{F}_q^\times}\theta(zb)+\sum_{y,z\in\mathbb{F}_q^\times}\theta(bz)\theta(-zy^2)+\sum_{x,z\in\mathbb{F}_q^\times}\theta(zx^d)\theta(zax)\theta(zb) \\ &+\sum_{x,y,z\in\mathbb{F}_q^\times}\theta(x^dz)\theta(axz)\theta(bz)\theta(-zy^2) \\ &= q^2+A+B+C+D. \end{aligned} \tag{3.6}$$

From (2.1), we find that $A=-1$. Applying Lemma 2.3 we have

$$\begin{aligned} B &= \sum_{y,z\in\mathbb{F}_q^\times}\theta(bz)\theta(-zy^2) \\ &= \frac{1}{(q-1)^2}\sum_{l,m=0}^{q-2}G_{-m}G_{-l}T^m(b)T^l(-1)\sum_{y\in\mathbb{F}_q^\times}T^{2l}(y)\sum_{z\in\mathbb{F}_q^\times}T^{l+m}(z). \end{aligned}$$

We now apply Lemma 2.1 to the inner sums on the right, which gives non zero sums only if $2l\equiv 0 \pmod{q-1}$ and $l+m\equiv 0 \pmod{q-1}$. Hence $l=0$ or $l=\frac{q-1}{2}$; and $m=0$ or $m=\frac{q-1}{2}$, respectively. Finally, using Lemma 2.2, we get $B=1+q\phi(b)$. Similarly,

$$\begin{aligned} D &= \sum_{x,y,z\in\mathbb{F}_q^\times}\theta(x^dz)\theta(axz)\theta(bz)\theta(-zy^2) \\ &= \frac{1}{(q-1)^4}\sum_{l,m,n,k=0}^{q-2}G_{-m}G_{-l}G_{-n}G_{-k}T^l(a)T^n(b)T^k(-1) \\ &\times\sum_{x\in\mathbb{F}_q^\times}T^{l+md}(x)\sum_{y\in\mathbb{F}_q^\times}T^{2k}(y)\sum_{z\in\mathbb{F}_q^\times}T^{l+m+n+k}(z). \end{aligned} \tag{3.7}$$

The inner sums are non zero only if $l+md\equiv 0 \pmod{q-1}$, $2k\equiv 0 \pmod{q-1}$, and $l+m+n+k\equiv 0 \pmod{q-1}$. This implies that $l\equiv -md \pmod{q-1}$, $k=0$ or $k=\frac{q-1}{2}$; and $n\equiv m(d-1) \pmod{q-1}$ or $n\equiv m(d-1)+\frac{q-1}{2} \pmod{q-1}$, respectively. Accounting for these values in (3.7), noting that $T^{q-1}=\varepsilon$, we obtain

$$\begin{aligned} D &= \frac{1}{q-1}\sum_{m=0}^{q-2}G_{-m}G_{md}G_{-m(d-1)}G_0T^{-md}(a)T^{m(d-1)}(b) \\ &+\frac{1}{q-1}\sum_{m=0}^{q-2}G_{-m}G_{md}G_{-m(d-1)+\frac{q-1}{2}}G_{\frac{q-1}{2}}T^{-md}(a)T^{m(d-1)+\frac{q-1}{2}}(b)T^{\frac{q-1}{2}}(-1) \\ &= \frac{-1}{q-1}\sum_{m=0}^{q-2}G_{-m}G_{md}G_{-m(d-1)}T^{-md}(a)T^{m(d-1)}(b) \\ &+\frac{\phi(-b)}{q-1}\sum_{m=0}^{q-2}G_{-m}G_{md}G_{-m(d-1)+\frac{q-1}{2}}G_{\frac{q-1}{2}}T^{-md}(a)T^{m(d-1)}(b). \end{aligned}$$

Expanding C in a similar fashion, using Lemma 2.3, it follows that the first term of the last expression for D will be equal to $-C$. Now substituting the expressions for A, B, C, and D in (3.6) we have

$$q\cdot N_d = q^2+q\phi(b)+\frac{\phi(-b)}{q-1}\sum_{m=0}^{q-2}G_{-m}G_{md}G_{-m(d-1)+\frac{q-1}{2}}G_{\frac{q-1}{2}}T^m\left(\frac{b^{d-1}}{a^d}\right).$$

$$= q^2+q\phi(b)+\frac{\phi(-1)}{q-1}\sum_{m=0}^{q-2}G_{-m+\frac{q-1}{2}}G_{md}G_{-m(d-1)}G_{\frac{q-1}{2}}T^m\left(\frac{b^{d-1}}{a^d}\right), \tag{3.8}$$

where we have replaced m by $m-\frac{q-1}{2}$ in the last sum. Using Davenport-Hasse relation (Theorem 2.4) for $k=2$ and $\psi=T^{-m}$ we deduce that

$$G_{-m+\frac{q-1}{2}} = \frac{G_{\frac{q-1}{2}}G_{-2m}T^m(4)}{G_{-m}}. \tag{3.9}$$

Substituting (3.9) into (3.8) and using Lemma 2.2 yields

$$q\cdot N_d = q^2+q\phi(b)+\frac{q}{q-1}+\frac{q\phi(b)}{q-1} + \frac{q}{q-1}\sum_{\substack{m=1\\ m\neq\frac{q-1}{2}}}^{q-2}\frac{G_{-2m}G_{md}G_{-m(d-1)}}{G_{-m}}T^m\left(\frac{4b^{d-1}}{a^d}\right).$$

Now we take T to be the inverse of the Teichmüller character, that is, $T=\overline{\omega}$, and then using Gross-Koblitz formula (Theorem 2.6) we deduce that

$$q\cdot N_d = q^2+q\phi(b)+\frac{q}{q-1}+\frac{q\phi(b)}{q-1}+\frac{q}{q-1}\sum_{\substack{m=1\\ m\neq\frac{q-1}{2}}}^{q-2}\pi^{(p-1)s}\,\overline{\omega}^m\left(\frac{4b^{d-1}}{a^d}\right) \times\prod_{i=0}^{r-1}\frac{\Gamma_p(\langle\frac{-2mp^i}{q-1}\rangle)\Gamma_p(\langle\frac{mdp^i}{q-1}\rangle)\Gamma_p(\langle\frac{-m(d-1)p^i}{q-1}\rangle)}{\Gamma_p(\langle\frac{-mp^i}{q-1}\rangle)},$$

where $s=\sum_{i=0}^{r-1}\left\{\langle\frac{-2mp^i}{q-1}\rangle+\langle\frac{mdp^i}{q-1}\rangle+\langle\frac{-m(d-1)p^i}{q-1}\rangle-\langle\frac{-mp^i}{q-1}\rangle\right\}$. Using Lemma 2.5, canceling q from both the sides, and rearranging the terms, we have

$$N_d = q+\phi(b)+\frac{1}{q-1}+\frac{\phi(b)}{q-1}+\frac{1}{q-1}\sum_{\substack{m=1\\ m\neq\frac{q-1}{2}}}^{q-2}\pi^{(p-1)s}\,\overline{\omega}^m\left(\frac{b^{d-1}d^d}{a^d(d-1)^{d-1}}\right) \times\prod_{i=0}^{r-1}\Gamma_p(\langle\frac{mp^i}{q-1}\rangle)\Gamma_p(\langle(1-\frac{m}{q-1})p^i\rangle)\frac{\Gamma_p(\langle(\frac{1}{2}-\frac{m}{q-1})p^i\rangle)\Gamma_p(\langle(\frac{1}{2}+\frac{m}{q-1})p^i\rangle)}{\Gamma_p(\langle\frac{p^i}{2}\rangle)\Gamma_p(\langle\frac{p^i}{2}\rangle)} \times\prod_{i=0}^{r-1}\frac{\prod_{h=0}^{d-3}\Gamma_p(\langle(\frac{1+h}{d-1}-\frac{m}{q-1})p^i\rangle)}{\prod_{h=1}^{d-2}\Gamma_p(\langle\frac{hp^i}{d-1}\rangle)}\prod_{\substack{h=1\\ h\neq\frac{d}{2}}}^{d-1}\frac{\Gamma_p(\langle(\frac{h}{d}+\frac{m}{q-1})p^i\rangle)}{\Gamma_p(\langle\frac{hp^i}{d}\rangle)}.$$

Applying Lemma 3.4 we then get

$$N_d = q+\phi(b)+\frac{1}{q-1}+\frac{\phi(b)}{q-1}+\frac{1}{q-1}\sum_{\substack{m=1\\ m\neq\frac{q-1}{2}}}^{q-2}(-1)^r\pi^{(p-1)s}\overline{\omega}^m\left(\frac{b^{d-1}d^d}{a^d(d-1)^{d-1}}\right)$$

$$\times\prod_{i=0}^{r-1}\prod_{h=1}^{d-2}\frac{\Gamma_p(\langle(\frac{h}{d-1}-\frac{m}{q-1})p^i\rangle)}{\Gamma_p(\langle\frac{hp^i}{d-1}\rangle)}\prod_{\substack{h=1\\ h\neq\frac{d}{2}}}^{d-1}\frac{\Gamma_p(\langle(\frac{h}{d}+\frac{m}{q-1})p^i\rangle)}{\Gamma_p(\langle\frac{hp^i}{d}\rangle)}. \tag{3.10}$$

Simplifying the term s we obtain $s=\sum_{i=0}^{r-1}\left\{\lfloor\frac{-mp^i}{q-1}\rfloor-\lfloor\frac{-2mp^i}{q-1}\rfloor-\lfloor\frac{mdp^i}{q-1}\rfloor-\lfloor\frac{-m(d-1)p^i}{q-1}\rfloor\right\}$, which is an integer. Plugging this expression in (3.10) we have

$$N_d = q+\phi(b)+\frac{1}{q-1}+\frac{\phi(b)}{q-1}+\frac{q}{q-1}\sum_{\substack{m=1\\ m\neq\frac{q-1}{2}}}^{q-2}\overline{\omega}^m\left(\frac{b^{d-1}d^d}{a^d(d-1)^{d-1}}\right)$$
$$\times(-p)^{\sum_{i=0}^{r-1}\left\{\lfloor\frac{-mp^i}{q-1}\rfloor-\lfloor\frac{-2mp^i}{q-1}\rfloor-\lfloor\frac{mdp^i}{q-1}\rfloor-\lfloor\frac{-m(d-1)p^i}{q-1}\rfloor-1\right\}}$$
$$\times\prod_{i=0}^{r-1}\prod_{h=1}^{d-2}\frac{\Gamma_p(\langle(\frac{h}{d-1}-\frac{m}{q-1})p^i\rangle)}{\Gamma_p(\langle\frac{hp^i}{d-1}\rangle)}\prod_{\substack{h=1\\ h\neq\frac{d}{2}}}^{d-1}\frac{\Gamma_p(\langle(\frac{h}{d}+\frac{m}{q-1})p^i\rangle)}{\Gamma_p(\langle\frac{hp^i}{d}\rangle)}.$$

Now using Lemma 3.1 we obtain

$$N_d = q+\phi(b)+\frac{1}{q-1}+\frac{\phi(b)}{q-1}+\frac{q}{q-1}\sum_{\substack{m=1\\ m\neq\frac{q-1}{2}}}^{q-2}\overline{\omega}^m\left(\frac{b^{d-1}d^d}{a^d(d-1)^{d-1}}\right)$$
$$\times\prod_{i=0}^{r-1}(-p)^{-\left\{\sum_{h=1}^{d-2}\lfloor\langle\frac{hp^i}{d-1}\rangle-\frac{mp^i}{q-1}\rfloor+\sum_{h=1,\ h\neq\frac{d}{2}}^{d-1}\lfloor\langle\frac{-hp^i}{d}\rangle+\frac{mp^i}{q-1}\rfloor\right\}}$$
$$\times\prod_{i=0}^{r-1}\prod_{h=1}^{d-2}\frac{\Gamma_p(\langle(\frac{h}{d-1}-\frac{m}{q-1})p^i\rangle)}{\Gamma_p(\langle\frac{hp^i}{d-1}\rangle)}\prod_{\substack{h=1\\ h\neq\frac{d}{2}}}^{d-1}\frac{\Gamma_p(\langle(\frac{h}{d}+\frac{m}{q-1})p^i\rangle)}{\Gamma_p(\langle\frac{hp^i}{d}\rangle)}. \tag{3.11}$$

Now for $m=0$ we have the following identities:

$$\sum_{h=1}^{d-2}\lfloor\langle\frac{hp^i}{d-1}\rangle-\frac{mp^i}{q-1}\rfloor+\sum_{h=1,\ h\neq\frac{d}{2}}^{d-1}\lfloor\langle\frac{-hp^i}{d}\rangle+\frac{mp^i}{q-1}\rfloor=0$$

and

$$\overline{\omega}^m\left(\frac{b^{d-1}d^d}{a^d(d-1)^{d-1}}\right)\prod_{i=0}^{r-1}\prod_{h=1}^{d-2}\frac{\Gamma_p(\langle(\frac{h}{d-1}-\frac{m}{q-1})p^i\rangle)}{\Gamma_p(\langle\frac{hp^i}{d-1}\rangle)}\prod_{\substack{h=1\\ h\neq\frac{d}{2}}}^{d-1}\frac{\Gamma_p(\langle(\frac{h}{d}+\frac{m}{q-1})p^i\rangle)}{\Gamma_p(\langle\frac{hp^i}{d}\rangle)}=1.$$

Also for $m=\frac{q-1}{2}$ we have

$$\sum_{h=1}^{d-2}\lfloor\langle\frac{hp^i}{d-1}\rangle-\frac{mp^i}{q-1}\rfloor+\sum_{h=1,\ h\neq\frac{d}{2}}^{d-1}\lfloor\langle\frac{-hp^i}{d}\rangle+\frac{mp^i}{q-1}\rfloor=0,$$

and by Lemma 2.5 we have

$$\overline{\omega}^m\left(\frac{b^{d-1}d^d}{a^d(d-1)^{d-1}}\right)\prod_{i=0}^{r-1}\prod_{h=1}^{d-2}\frac{\Gamma_p(\langle(\frac{h}{d-1}-\frac{m}{q-1})p^i\rangle)}{\Gamma_p(\langle\frac{hp^i}{d-1}\rangle)}\prod_{\substack{h=1\\h\neq\frac{d}{2}}}^{d-1}\frac{\Gamma_p(\langle(\frac{h}{d}+\frac{m}{q-1})p^i\rangle)}{\Gamma_p(\langle\frac{hp^i}{d}\rangle)}=\phi(b).$$

Using all these four identities in (3.11) we deduce that

$$\begin{aligned}
N_d &= q-1+\frac{q}{q-1}\sum_{m=0}^{q-2}\overline{\omega}^m\left(\frac{b^{d-1}d^d}{a^d(d-1)^{d-1}}\right)\\
&\times\prod_{i=0}^{r-1}(-p)^{-\left\{\sum_{h=1}^{d-2}\lfloor\langle\frac{hp^i}{d-1}\rangle-\frac{mp^i}{q-1}\rfloor+\sum_{h=1,\ h\neq\frac{d}{2}}^{d-1}\lfloor\langle\frac{-hp^i}{d}\rangle+\frac{mp^i}{q-1}\rfloor\right\}}\\
&\times\prod_{i=0}^{r-1}\prod_{h=1}^{d-2}\frac{\Gamma_p(\langle(\frac{h}{d-1}-\frac{m}{q-1})p^i\rangle)}{\Gamma_p(\langle\frac{hp^i}{d-1}\rangle)}\prod_{\substack{h=1\\h\neq\frac{d}{2}}}^{d-1}\frac{\Gamma_p(\langle(\frac{h}{d}+\frac{m}{q-1})p^i\rangle)}{\Gamma_p(\langle\frac{hp^i}{d}\rangle)}\\
&= q-1-q\\
&\times {}_{d-2}G_{d-2}\left[\begin{array}{cccccccc}\frac{1}{d-1}, & \frac{2}{d-1}, & \ldots, & \frac{d-2}{2(d-1)}, & \frac{d}{2(d-1)}, & \ldots, & \frac{d-2}{d-1} & \\ \frac{1}{d}, & \frac{2}{d}, & \ldots, & \frac{d-2}{2d}, & \frac{d+2}{2d}, & \ldots, & \frac{d-1}{d} & \end{array}|f(0)\right]_q,
\end{aligned}$$

where we have used the fact that

$$\prod_{\substack{h=1\\h\neq\frac{d}{2}}}^{d-1}\frac{\Gamma_p(\langle(\frac{h}{d}+\frac{m}{q-1})p^i\rangle)}{\Gamma_p(\langle\frac{hp^i}{d}\rangle)}=\prod_{\substack{h=1\\h\neq\frac{d}{2}}}^{d-1}\frac{\Gamma_p(\langle(\frac{-h}{d}+\frac{m}{q-1})p^i\rangle)}{\Gamma_p(\langle\frac{-hp^i}{d}\rangle)}.$$

□

Theorem 3.7. *Let $d\geq 3$ be odd, and let p be an odd prime such that $p\nmid d(d-1)$. Then*

$$\begin{aligned}
N_d &= q-q\phi(-ab)\\
&\times {}_{d-1}G_{d-1}\left[\begin{array}{cccccccc}\frac{1}{2(d-1)}, & \frac{3}{2(d-1)}, & \ldots, & \frac{d-2}{2(d-1)}, & \frac{d}{2(d-1)}, & \ldots, & \frac{2d-3}{2(d-1)} & \\ \frac{1}{2d}, & \frac{3}{2d}, & \ldots, & \frac{d-2}{2d}, & \frac{d+2}{2d}, & \ldots, & \frac{2d-1}{2d} & \end{array}|-f(0)\right]_q,
\end{aligned}$$

where f is defined as in Theorem 1.2.

Proof. Following the proof of Theorem 3.6 we have

$$\begin{aligned}
q\cdot N_d &= q^2+q\phi(b)+\frac{\phi(-ab)}{q-1}\sum_{m=0}^{q-2}G_{-m+\frac{q-1}{2}}G_{md+\frac{q-1}{2}}G_{-m(d-1)+\frac{q-1}{2}}\\
&\times G_{\frac{q-1}{2}}T^m\left(\frac{b^{d-1}}{a^d}\right).
\end{aligned}\tag{3.12}$$

Using Davenport-Hasse relation (Theorem 2.4) for $k=2$ and $\psi=T^{-m},T^{md}$, and $T^{-m(d-1)}$, and Lemma 2.2, in (3.12) we obtain

$$N_d=q+\phi(b)+\frac{\phi(b)}{q-1}+\frac{q\phi(-ab)}{q-1}\sum_{\substack{m=0\\m\neq\frac{q-1}{2}}}^{q-2}\frac{G_{-2m}G_{2md}G_{-2m(d-1)}}{G_{-m}G_{md}G_{-m(d-1)}}T^m\left(\frac{b^{d-1}}{a^d}\right).$$

Letting $T = \overline{\omega}$, and using the Gross-Koblitz formula (Theorem 2.6) we get that

$$N_d = q + \phi(b) + \frac{\phi(b)}{q-1} + \frac{q\phi(-ab)}{q-1} \sum_{\substack{m=0 \\ m \neq \frac{q-1}{2}}}^{q-2} \pi^{(p-1)s}\, \overline{\omega}^m \left(\frac{b^{d-1}}{a^d}\right)$$
$$\times \prod_{i=0}^{r-1} \frac{\Gamma_p(\langle \frac{-2mp^i}{q-1} \rangle)\Gamma_p(\langle \frac{2mdp^i}{q-1} \rangle)\Gamma_p(\langle \frac{-2m(d-1)p^i}{q-1} \rangle)}{\Gamma_p(\langle \frac{-mp^i}{q-1} \rangle)\Gamma_p(\langle \frac{mdp^i}{q-1} \rangle)\Gamma_p(\langle \frac{-m(d-1)p^i}{q-1} \rangle)},$$

where

$$s = \sum_{i=0}^{r-1} \left\{ \lfloor \frac{-mp^i}{q-1} \rfloor + \lfloor \frac{mdp^i}{q-1} \rfloor + \lfloor \frac{-m(d-1)p^i}{q-1} \rfloor \right\}$$
$$- \sum_{i=0}^{r-1} \left\{ \lfloor \frac{-2mp^i}{q-1} \rfloor + \lfloor \frac{2mdp^i}{q-1} \rfloor + \lfloor \frac{-2m(d-1)p^i}{q-1} \rfloor \right\},$$

which is an integer. Using Lemma 2.5 and Lemma 3.4 we deduce that

$$N_d = q + \phi(b) + \frac{\phi(b)}{q-1} + \frac{q\phi(-ab)}{q-1} \sum_{\substack{m=0 \\ m \neq \frac{q-1}{2}}}^{q-2} \pi^{(p-1)s}\overline{\omega}^m \left(\frac{-b^{d-1}d^d}{a^d(d-1)^{d-1}}\right)$$
$$\times \prod_{i=0}^{r-1} \prod_{\substack{h=1 \\ h \text{ odd}}}^{2d-3} \frac{\Gamma_p(\langle (\frac{h}{2(d-1)} - \frac{m}{q-1})p^i \rangle)}{\Gamma_p(\langle \frac{hp^i}{2(d-1)} \rangle)} \times \prod_{\substack{h=1 \\ h \text{ odd} \\ h \neq d}}^{2d-1} \frac{\Gamma_p(\langle (\frac{-h}{2d} + \frac{m}{q-1})p^i \rangle}{\Gamma_p(\langle \frac{-hp^i}{2d} \rangle)}.$$

We now calculate the term under summation for $m = \frac{q-1}{2}$ separately using Lemma 2.5 and Lemma 3.3, and deduce the required expression. □

Theorem 3.8. *Let $d \geq 4$ be even, and let p be an odd prime such that $p \nmid d(d-1)$. Then*

$$N'_d = q - 1 - q\phi(b)$$
$$\times {}_{d-2}G_{d-2} \left[\begin{matrix} \frac{1}{d-1}, & \frac{2}{d-1}, & \ldots, & \frac{d-2}{2(d-1)}, & \frac{d}{2(d-1)}, & \ldots, & \frac{d-2}{(d-1)} \\ \frac{1}{d}, & \frac{2}{d}, & \ldots, & \frac{d-2}{2d}, & \frac{d+2}{2d}, & \ldots, & \frac{d-1}{d} \end{matrix} \middle| g(0) \right]_q,$$

where g is defined as in Theorem 1.2.

Proof. The proof proceeds along similar lines to the proofs of Theorems 3.6 and 3.7 so we omit the details for reasons of brevity.

□

Theorem 3.9. *Let $d \geq 3$ be odd, and let p be an odd prime such that $p \nmid d(d-1)$. Then*

$$N'_d = q - q\phi(b)$$
$$\times {}_{d-1}G_{d-1} \left[\begin{matrix} \frac{1}{d-1}, & \frac{2}{d-1}, & \ldots, & \frac{d-1}{2(d-1)}, & \frac{d+1}{2(d-1)}, & \ldots, & \frac{d-2}{d-1}, & \frac{1}{2} \\ \frac{1}{d}, & \frac{2}{d}, & \ldots, & \frac{d-1}{2d}, & \frac{d+1}{2d}, & \ldots, & \frac{d-2}{d}, & \frac{d-1}{d} \end{matrix} \middle| -g(0) \right]_q,$$

where g is defined as in Theorem 1.2.

Proof. The proof proceeds along similar lines to the proofs of Theorems 3.6 and 3.7 so we omit the details for reasons of brevity. However, we apply Lemma 3.2 to deduce the final expression. $\square$

Remark 3.10. Putting $d = 3$ in Theorem 3.9 we can derive [5, Theorem 3.4].

Proof of Theorem 1.2: Consider the hyperelliptic curves $E_d : y^2 = x^d + ax + b$ and $E'_d : y^2 = x^d + ax^{d-1} + b$, where $a, b \neq 0$. We have

$$N_d = \#\{(x,y) \in \mathbb{F}_q^2 : x^d + ax + b - y^2 = 0\} = \sum_{y \in \mathbb{F}_q} N(x^d + ax + b - y^2 = 0), \tag{3.13}$$

where, for a given y, $N(x^d + ax + b - y^2 = 0)$ denotes the number of distinct zeros of the polynomial $x^d + ax + b - y^2$. If b is not a square in $\mathbb{F}_q$ then the term $b - y^2 \neq 0$ for all $y \in \mathbb{F}_q$. Applying Theorem 1.7 we have

$$N(x^d + ax + b - y^2 = 0) = 1 + \phi(y^2 - b)$$
$$\times {}_{d-1}G_{d-1}\left[\begin{matrix} \frac{1}{2(d-1)}, & \frac{3}{2(d-1)}, & \ldots, & \frac{d-1}{2(d-1)}, & \frac{d+1}{2(d-1)}, & \ldots, & \frac{2d-3}{2(d-1)} \\ 0, & \frac{1}{d}, & \ldots, & \frac{\frac{d}{2}-1}{d}, & \frac{\frac{d}{2}+1}{d}, & \ldots, & \frac{d-1}{d} \end{matrix} \Big| f(y)\right]_q,$$

where $f(y) = \frac{d}{a}\left(\frac{(b-y^2)d}{a(d-1)}\right)^{d-1}$. Now putting the value of $N(x^d + ax + b - y^2 = 0)$ in (3.13), and then applying Theorem 3.6 we easily derive the first summation identity. To derive the second summation identity we consider the hyperelliptic curve E'_d and the proof is similar to that of the first summation identity. If b is not a square in $\mathbb{F}_q$, using Theorem 1.9 and Theorem 3.8 we derive the second summation identity.

If b is a square in $\mathbb{F}_q$, then for $y = \pm\sqrt{b}$ the term $b - y^2 = 0$. Hence

$$\begin{aligned} N_d &= \#\{(x,y) \in \mathbb{F}_q^2 : x^d + ax + b - y^2 = 0\} \\ &= \sum_{\substack{y \in \mathbb{F}_q \\ y \neq \pm\sqrt{b}}} N(x^d + ax + b - y^2 = 0) + 2 \cdot N(x(x^{d-1} + a) = 0). \end{aligned}$$

Using Lemma 3.5 we have $N(x(x^{d-1} + a) = 0) = 1 + \sum_{j=0}^{l-1} \chi^j(-a)$, where $l = \gcd(d-1, q-1)$ and χ is a character of order l. Thus,

$$N_d = 2 + 2 \cdot \sum_{j=0}^{l-1} \chi^j(-a) + \sum_{\substack{y \in \mathbb{F}_q \\ y \neq \pm\sqrt{b}}} N(x^d + ax + b - y^2 = 0). \tag{3.14}$$

Now applying Theorem 1.7 and Theorem 3.6 in (3.14), we deduce the third summation identity. Again, if b is a square then

$$\begin{aligned} N'_d &= \#\{(x,y) \in \mathbb{F}_q^2 : x^d + ax^{d-1} + b - y^2 = 0\} \\ &= \sum_{\substack{y \in \mathbb{F}_q \\ y \neq \pm\sqrt{b}}} N(x^d + ax^{d-1} + b - y^2 = 0) + 2 \cdot N(x^{d-1}(x + a) = 0) \\ &= \sum_{\substack{y \in \mathbb{F}_q \\ y \neq \pm\sqrt{b}}} N(x^d + ax^{d-1} + b - y^2 = 0) + 4. \end{aligned} \tag{3.15}$$

Now applying Theorem 1.9 and Theorem 3.8 in (3.15) we derive the fourth summation identity of the theorem. This completes the proof of the theorem.

Proof of Theorem 1.3: Here d is odd. Following the proof of Theorem 1.2 and applying Theorem 3.7, Theorem 3.9, Theorem 1.8, and Theorem 1.10, we can derive all the four summation identities. This completes the proof of the theorem.

Proof of Theorem 1.5: In [19, Theorem 1.2], the third author gave a formula for the number of points on the elliptic curve $y^2 = x^3 + ax + b$ as a special value of ${}_2G_2[\cdots]_p$ when the j-invariant of the curve is different from 0 and 1728. We have verified that the result is also true for $\mathbb{F}_q$. Thus, from [19, Theorem 1.2] we have

$$\#\{(x,y) \in \mathbb{F}_q^2 : y^2 = x^3 + ax + b\} = q - \phi(b) \cdot q \cdot {}_2G_2\left[\begin{array}{cc} \frac{1}{4}, & \frac{3}{4} \\ \frac{1}{3}, & \frac{2}{3} \end{array} | \frac{-27b^2}{4a^3}\right]_q,$$

where $a, b \neq 0$ and $\dfrac{-27b^2}{4a^3} \neq 1$. Now taking $d = 3$ in Theorem 3.7, we have

$$\#\{(x,y) \in \mathbb{F}_q^2 : y^2 = x^3 + ax + b\} = q - q \cdot \phi(-ab) \cdot {}_2G_2\left[\begin{array}{cc} \frac{1}{4}, & \frac{3}{4} \\ \frac{1}{6}, & \frac{5}{6} \end{array} | \frac{-27b^2}{4a^3}\right]_q,$$

where $a, b \neq 0$. Comparing both the identities completes the proof.

4. SPECIAL VALUES OF ${}_nG_n[\cdots]$ FOR $n = 2, 3, 4$

Finding special values of hypergeometric function is an important and interesting problem. Many special values of hypergeometric functions over finite fields are obtained (see for example [1, 3, 8, 20]). These results can be re-written in terms of ${}_2G_2[\cdots]$ and ${}_3G_3[\cdots]$. However, no special value of ${}_nG_n[\cdots]$ is obtained in full generality to date. In [4], the first and second author expressed the number of distinct zeros of the polynomials $x^d + ax + b$ and $x^d + ax^{d-1} + b$ over $\mathbb{F}_q$ in terms of values of the function ${}_{d-1}G_{d-1}[\cdots]$. We now look at those expressions more closely and derive certain special values of the function ${}_nG_n[\cdots]$ when $n = 2, 3, 4$.

Theorem 4.1. *Let $a, b, c \in \mathbb{F}_q^\times$ be such that $a + b + c = 0$ and $ab + bc + ca \neq 0$. Then, for $p \geq 5$, we have*

$$(4.1) \qquad {}_2G_2\left[\begin{array}{cc} 0, & \frac{1}{2} \\ \frac{1}{6}, & \frac{5}{6}, \end{array} | - \frac{27a^2b^2c^2}{4(ab+bc+ca)^3}\right]_q = A \cdot \phi(-(ab+bc+ca)),$$

where $A = 2$ if all of a, b, c are distinct and $A = 1$ if exactly two of a, b, c are equal.

If $a, b, c \in \mathbb{F}_q^\times$ are such that $ab + bc + ca = 0$ and $a + b + c \neq 0$. Then, for $p \geq 5$, we have

$$(4.2) \qquad {}_2G_2\left[\begin{array}{cc} 0, & \frac{1}{2} \\ \frac{1}{6}, & \frac{5}{6}, \end{array} | - \frac{27abc}{4(a+b+c)^3}\right]_q = A \cdot \phi(-abc(a+b+c)).$$

Proof. We have $(x-a)(x-b)(x-c) = x^3 - (a+b+c)x^2 + (ab+bc+ca)x - abc$. Now if $a + b + c = 0$ and $ab + bc + ca \neq 0$, then putting $d = 3$ in Theorem 1.8, we can easily deduce (4.1). Similarly, (4.2) follows from Theorem 1.10. $\square$

Example 4.2. Put $a = b = 1$ and $c = -2$ in (4.1), then for all $p \geq 5$, we have

$${}_2G_2\left[\begin{array}{cc} 0, & \frac{1}{2} \\ \frac{1}{6}, & \frac{5}{6}, \end{array} |1\right]_q = \phi(3).$$

Theorem 4.3. *If $p \geq 5$ then we have*

$$ {}_3G_3\left[\begin{array}{ccc} \frac{1}{6}, & \frac{1}{2}, & \frac{5}{6} \\ 0, & \frac{1}{4}, & \frac{3}{4} \end{array} |1\right]_q = \phi(-3) + \phi(6). $$

Proof. We have

$$ x^4 - \frac{a^3}{2}x + \frac{3a^4}{16} = (x - \frac{a}{2})^2(x^2 + ax + \frac{3a^2}{4}). $$

Hence

$$ N(x^4 - \frac{a^3}{2}x + \frac{3a^4}{16} = 0) = 1 + N(x^2 + ax + \frac{3a^2}{4} = 0) = 2 + \phi(-2). \tag{4.3} $$

Now applying Theorem 1.7 on the left side of (4.3) we obtain the result. □

Theorem 4.4. *If $p > 7$ and $p \neq 23$, then*

$$ {}_4G_4\left[\begin{array}{cccc} 0, & \frac{1}{4}, & \frac{1}{2}, & \frac{3}{4} \\ \frac{1}{10}, & \frac{3}{10}, & \frac{7}{10}, & \frac{9}{10} \end{array} | -\frac{5^5}{4^4}\right]_q = \phi(-1) + \phi(3) + \phi(-1) \cdot {}_2G_2\left[\begin{array}{cc} 0, & \frac{1}{2} \\ \frac{1}{6}, & \frac{5}{6}, \end{array} |\frac{27}{4}\right]_q. $$

Proof. We have $x^5+ax^4+a^5 = (x^3-a^2x+a^3)(x^2+ax+a^2)$. Let $f(x) = x^5+ax^4+a^5$. Then $f'(x) = 5x^4 + 4ax^3$. If $f(x)$ has a repeated zero, say c, then $f(c) = 0$ and $f'(c) = 0$. Solving these two equations, we have $3381 = 3.7^2.23 = 0$ in $\mathbb{F}_q$. Hence, if $p \neq 3, 7, 23$, we have

$$ N(x^5 + ax^4 + a^5 = 0) = N(x^3 - a^2x + a^3 = 0) + N(x^2 + ax + a^2 = 0). $$

Now the result easily follows from Theorem 1.10 and Theorem 1.8. □

5. CONCLUDING REMARKS

The technique used to derive the summation identities for ${}_nG_n[\cdots]_q$ in this paper and for ${}_2G_2[\cdots]_q$ in [5, 6] is based on counting points on families of certain algebraic varieties and counting zeros on certain families of polynomials. This technique is quite involved. Also, in finding the special values of the function ${}_nG_n[\cdots]$ when $n = 2, 3, 4$, we factored the polynomials $x^d + ax + b$ and $x^d + ax^{d-1} + b$ into polynomials of the same form of lower degree when $d = 5, 4, 3$. However, such factorizations do not exist when $d > 5$. Hence, our method can't be applied to deduce special values of ${}_nG_n[\cdots]$ when $n \geq 5$. Therefore, it would be very beneficial if we could derive these summation identities for ${}_nG_n[\cdots]$ and special values more directly using properties of p-adic gamma function.

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