EIGENVALUES AND FOURIER COEFFICIENTS OF DEGREE TWO SIEGEL EIGENFORMS CONSTRUCTED FROM IGUSA THETA CONSTANTS

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Abstract. We prove multiplicative relations among certain Fourier coefficients of degree two Siegel eigenforms constructed from Igusa theta constants with half-integral characteristics. We also provide simple relations between their eigenvalues and their Fourier coefficients.

1. Introduction

One of the most important questions relating to modular forms is that of multiplicity one, which asks, is a Hecke eigenform uniquely determined (up to scalar multiple) by its eigenvalues. A closely related question is, given an eigenform's eigenvalues, is it possible to explicitly determine its Fourier coefficients. In the case of elliptic modular forms, we can answer both questions in the affirmative (after accounting for the theory of newforms) and we have simple relations linking the eigenvalues to the Fourier coefficients, as well as multiplicative relations among the coefficients, allowing us to reconstruct an eigenform from its eigenvalues.

In the case of Siegel modular forms in general, the picture is less clear. In a recent breakthrough, Schmidt [10] provides a multiplicity one result for degree two Siegel eigenforms on the full modular group. In particular, if F_1, F_2 are cuspidal Hecke eigenforms of weight k, with eigenvalues $\lambda_1(\cdot), \lambda_2(\cdot)$, respectively, and, for almost all primes p, $\lambda_1(p^r) = \lambda_2(p^r)$ for $r \in \{1, 2\}$, then F_1 is a scalar multiple of F_2. In fact, Schmidt proves a stronger result, where F_1 and F_2 are not assumed to have the same weight, and these results form part of a more general result establishing multiplicity one for paramodular cusp forms. More recently, Schmidt's result has been strengthened to hold if the eigenvalues agree at a set of primes of positive upper density, by Kumar, Meher and Shankhadhar [8].

However, it is, as yet, unclear as to whether it is possible to explicitly reconstruct a Siegel eigenform's Fourier coefficients from its eigenvalues. A modest step in this direction by this author in [9], following the work of Andrianov [1], provides simple relations between the eigenvalues and the Fourier coefficients, and also multiplicative relations among the Fourier coefficients, in certain cases. Specifically, let $F(Z) = \sum_{N \geq 0} \mathfrak{a}(N) \exp(2\pi i \operatorname{Tr}(NZ))$ be a degree two weight k Hecke eigenform on the full modular group, normalized with $\mathfrak{a}(I) = 1$ (where I is the identity matrix), with eigenvalues $\lambda(\cdot)$. Then, for any odd prime p,

$$\lambda(p) = \mathfrak{a}(pI) + \left(1 + \left(\tfrac{-1}{p}\right)\right) p^{k-2}$$

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and

$$\lambda(p^2) = \mathfrak{a}(p^2 I) + \left(1 + \left(\tfrac{-1}{p}\right)\right)\left(p^{k-2}\,\mathfrak{a}(pI) + p^{2k-4}\right),$$

where $\left(\frac{\cdot}{p}\right)$ is the Legendre symbol modulo p. Also, for $m, n \in \mathbb{Z}^+$ with $\gcd(m,n) = 1$,

$$\mathfrak{a}\,(mnI) = \mathfrak{a}\,(mI)\;\mathfrak{a}\,(nI)\,.$$

Relations for $\mathfrak{a}(p^{r+1}I)$ in terms of $\mathfrak{a}(p^r I)$ and $\mathfrak{a}(p^{r-1}I)$, $r \in \mathbb{Z}^+$, are also provided. These results were subsequently extended to eigenforms on $\Gamma_0(N)$ by Walling [11]. However, Siegel modular forms constructed from Igusa theta constants, which transform on the so-called theta group and are an important class of forms, are not covered by the above results. The purpose of this paper is to prove similar relations between the eigenvalues and the Fourier coefficients, and also multiplicative relations among the Fourier coefficients, of Siegel eigenforms constructed from Igusa theta constants with half-integral characteristics. We use similar methods to those in [9]. However, the approach is less strighforward in this case, as certain properties of the eigenform very much depend on the characteristics of theta constants.

Before stating our results, we will need to give some background on Igusa theta constants. In the next section we outline some notation. Section 3 gives details of Siegel modular forms on principal congruence subgroups and the associated Hecke theory. In Section 4, we introduce Igusa theta constants, outline some of their important properties and state our results. Our main results appear in Theorems 4.4 and 4.5. We then go on to prove these results in Section 5.

2. Notation

Let $\mathbb{A}^{m\times n}$ denote the set of all $m \times n$ matrices with entries in the set $\mathbb{A}$. For a matrix M we let ${}^tM$ denote its transpose; if M is square, $\mathrm{Tr}(M)$ its trace and $\mathrm{Det}(M)$ its determinant; and if M has entries in $\mathbb{C}$, $\mathrm{Im}(M)$ its imaginary part. For a square matrix M, we can form a vector of its diagonal entries, arranged in a natural way, which we denote $\mathrm{diag}(M)$. Also, for an $n \times n$ matrix M, we let $\mathrm{ATr}(M)$ be the sum of the entries on its anti-diagonal, i.e., if $M = (m_{i,j})$ then $\mathrm{ATr}(M) = \sum_{i=1}^{n} m_{i,n+1-i}$. If a matrix $M \in \mathbb{R}^{n\times n}$ is positive definite, then we write $M > 0$, and if M is positive semi-definite, we write $M \geq 0$. If M is a square matrix we set $\mathbf{e}\{M\} := \exp(\pi i\,\mathrm{Tr}(M))$. We denote the $n \times n$ identity matrix by I_n. Throughout this paper, we will often drop subscripts and/or superscripts from notation when the size/degree is clear from the context.

3. Siegel Modular Forms and Hecke Operators

We start with a brief summary of Siegel modular forms. See [2] for more details. The Siegel half-plane $\mathbb{H}^g$ of degree g is defined by

$$\mathbb{H}^g := \left\{ Z \in \mathbb{C}^{g\times g} \mid {}^tZ = Z, \mathrm{Im}(Z) > 0 \right\}.$$

Let

$$\Gamma^g := \mathrm{Sp}_{2g}(\mathbb{Z}) = \left\{ \mathcal{M} \in \mathbb{Z}^{2g\times 2g} \mid {}^t\mathcal{M} J \mathcal{M} = J \right\}, \qquad J = \left(\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right),$$

be the Siegel (full) modular group of degree g and let

$$\Gamma^g(q) := \{ \mathcal{M} \in \Gamma^g \mid \mathcal{M} \equiv I_{2g} \pmod{q} \}$$

be its principal congruence subgroup of level $q \in \mathbb{Z}^+$. If Γ' is a subgroup of Γ^g such that $\Gamma^g(q) \subset \Gamma'$ for some minimal q, then we say Γ' is a congruence subgroup of degree g and level q. The modular group Γ^g acts on $\mathbb{H}^g$ via the operation

$$\mathcal{M} \cdot Z = (AZ + B)\,(CZ + D)^{-1}\,,$$

where $\mathcal{M} = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma^g$, $Z \in \mathbb{H}^g$. Let Γ' be a congruence subgroup of degree g and level q, and let χ be a character on Γ'. A holomorphic function $F : \mathbb{H}^g \to \mathbb{C}$ is called a Siegel modular form of degree g, weight $k \in \mathbb{Z}^+$ and level q on Γ' if

$$F|_k\mathcal{M}(Z) := \mathrm{Det}(CZ + D)^{-k}\, F(\mathcal{M} \cdot Z) = \chi(\mathcal{M})F(Z)$$

for all $\mathcal{M} = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma'$. When $g = 1$ we also require the usual growth condition. The set of all such modular forms is a finite dimensional vector space over $\mathbb{C}$, which we denote $\mathfrak{M}_k(\Gamma', \chi)$. Every $F \in \mathfrak{M}_k(\Gamma', \chi)$ has a Fourier expansion of the form

$$F(Z) = \sum_{N \in \mathcal{R}^g} \mathfrak{a}(N) \exp\left(\tfrac{2\pi i}{q'}\, \mathrm{Tr}(NZ)\right)$$

where q' is a positive integer depending on Γ' and χ, and

$$\mathcal{R}^g = \left\{N = (N_{ij}) \in \mathbb{Q}^{g\times g} \mid {}^tN = N \geq 0; N_{ii}, 2N_{ij} \in \mathbb{Z}\right\}.$$

If $\Gamma' = \Gamma^g(q)$ and χ is trivial then $q' = q$. We note that

$$\mathfrak{a}(UN^tU) = \mathrm{Det}(U)^k\, \chi(\mathcal{M})\, \mathfrak{a}(N) \tag{3.1}$$

for all $\mathcal{M} = \left(\begin{smallmatrix} {}^tU^{-1} & 0 \\ 0 & U \end{smallmatrix}\right) \in \Gamma'$. We call $F \in \mathfrak{M}_k(\Gamma', \chi)$ a cusp form if $\mathfrak{a}(N) = 0$ for all $N \not> 0$.

In [4], Evdokimov gives a very nice description of the Hecke theory for Siegel modular forms on the principal congruence subgroup. We gave a brief summary here. Let

$$S^{(g)} := \left\{\mathcal{M} \in \mathbb{Z}^{2g\times 2g} \mid {}^t\mathcal{M}J\mathcal{M} = \mu(\mathcal{M})J,\ \mu(\mathcal{M}) = 1, 2, \cdots\right\}$$

and

$$S^{(g)}(q) := \left\{\mathcal{M} \in S^{(g)} \mid \mathcal{M} \equiv \begin{pmatrix} I & 0 \\ 0 & \mu(\mathcal{M})I \end{pmatrix} \pmod{q}, \gcd(\mu(\mathcal{M}), q) = 1\right\}.$$

Then every double coset $\Gamma^g(q)\mathcal{M}\Gamma^g(q)$, with $\mathcal{M} \in S^{(g)}(q)$, can be written as union of finitely many right cosets of $\Gamma^g(q)$ in $S^{(g)}(q)$, i.e.,

$$\Gamma^g(q)\mathcal{M}\Gamma^g(q) = \bigcup_{i=1}^{\nu} \Gamma^g(q)\, \sigma_i,$$

for some $\sigma_i \in S^{(g)}(q)$, $\nu \in \mathbb{Z}^+$. For each such double coset we associate an operator $T_k(\Gamma^g(q)\mathcal{M}\Gamma^g(q))$ which acts on $\mathfrak{M}_k(\Gamma^g(q))$ as follows. For $F \in \mathfrak{M}_k(\Gamma^g(q))$,

$$T_k(\Gamma^g(q)\mathcal{M}\Gamma^g(q))F := \mu(\mathcal{M})^{gk - \frac{g(g+1)}{2}} \sum_{i=1}^{\nu} F|_k\sigma_i.$$

$T_k(\Gamma^g(q)\mathcal{M}\Gamma^g(q))$ is independent of the choice of representatives $\{\sigma_i\}$ and maps $\mathfrak{M}_k(\Gamma^g(q))$ into itself. We call $F \in \mathfrak{M}_k(\Gamma^g(q))$ an eigenform if it is an eigenfunction for all the operators $T_k(\Gamma^g(q)\mathcal{M}\Gamma^g(q))$, $\mathcal{M} \in S^{(g)}(q)$. For all $k, g \geq 1$, $\mathfrak{M}_k(\Gamma^g(q))$ has a basis consisting of eigenforms. For $\gcd(m, q) = 1$, we define the Hecke operator of index m, $T_k(m)$, by the following finite sum:

$$T_k(m) := \sum_{\mu(\mathcal{M})=m} T_k(\Gamma^g(q)\mathcal{M}\Gamma^g(q)).$$

Then

$$T_k(m)\, T_k(n) = T_k(n)\, T_k(m) = T_k(mn), \quad \text{when } (m,n)=1. \tag{3.2}$$

For $F \in \mathfrak{M}_k(\Gamma^g(q))$ an eigenform, we define its eigenvalues, $\lambda_F(m)$, for $\gcd(m, q) = 1$, by

$$T_k(m)\, F = \lambda_F(m)\, F.$$

We will refer to $\lambda_F(m)$ as the eigenvalue of index m associated to F.

Let $V_i(m) \in \mathbb{Z}^{g\times g}$ be the diagonal matrix whose first i diagonal entries equal m and whose remaining diagonal entries equal 1. Now let $\mathcal{M}_i(m) \in \Gamma^g$ such that $\mathcal{M}_i(m) \equiv \left(\begin{smallmatrix} {}^tV_i(m)^{-1} & 0 \\ 0 & V_i(m)\end{smallmatrix}\right) \pmod q$. For $\chi_1, \chi_2, \ldots, \chi_g$ multiplicative characters modulo q, we define

$$\begin{aligned}&\mathfrak{M}_k(\Gamma^g(q); \chi_1, \chi_2, \ldots, \chi_g)\\ &\quad := \{F \in \mathfrak{M}_k(\Gamma^g(q)) \mid F|_k\mathcal{M}_i(m) = \chi_i(m)F \text{ for all } i \in \{1, 2, \ldots, g\}, \gcd(m, q) = 1\}\end{aligned}$$

The space $\mathfrak{M}_k(\Gamma^g(q); \chi_1, \chi_2, \ldots, \chi_g)$ is invariant under the action of the Hecke operators and we have the direct sum decomposition [4, p. 437]

$$\mathfrak{M}_k(\Gamma^g(q)) = \bigoplus_{\chi_1,\chi_2,\ldots,\chi_g} \mathfrak{M}_k(\Gamma^g(q); \chi_1, \chi_2, \ldots, \chi_g)$$

where the sum is over all sets of g multiplicative characters modulo q.

Let $F(Z) = \sum_{N\in\mathcal{R}^2} \mathfrak{a}(N) \exp\left(\frac{2\pi i}{q}\operatorname{Tr}(NZ)\right) \in \mathfrak{M}_k(\Gamma^2(q); \chi_1, \chi_2)$. In [4], Evdokimov considers the Fourier coefficients of $T_k(m)\, F$, where $\gcd(m, q) = 1$. Given (3.2), it suffices to study $T_k(p^\delta)\, F$, for p prime with $\gcd(p, q) = 1$ and $\delta \geq 1$. Let

$$T_k(p^\delta)\, F(Z) = \sum_{N\in\mathcal{R}^2} \mathfrak{a}(p^\delta; N) \exp\left(\frac{2\pi i}{q}\operatorname{Tr}(NZ)\right).$$

Evdokimov provides us with a formula for $\mathfrak{a}(p^\delta; N)$ in terms of $\mathfrak{a}(\cdot)$, the Fourier coefficients of F, which we state in Theorem 3.1 below. We first note that if F is an eigenform, then for any $N \in \mathcal{R}^2$ we have the relation

$$\mathfrak{a}(N)\, \lambda(p^\delta) = \mathfrak{a}(p^\delta; N). \tag{3.3}$$

Let

$$R(p^\beta) = \left\{ \begin{pmatrix} u_1 & u_2 \\ u_3 & u_4\end{pmatrix} \in SL_2(\mathbb{Z}) \mid (u_1, u_2) \pmod{p^\beta} \right\}$$

be any set of 2×2 integral matrices whose first row ranges over a complete set of representatives of the equivalence classes of relatively prime integers under the equivalence relation

$$(u_1, u_2) \sim (u_1', u_2') \pmod{p^\beta} \Leftrightarrow lu_1 \equiv u_1', lu_2 \equiv u_2' \pmod{p^\beta}, \tag{3.4}$$

for some $l \in \left(\mathbb{Z}/p^\beta\mathbb{Z}\right)^\times$, and whose second rows are chosen so that $u_1u_4 - u_2u_3 = 1$. We note that, if $\gcd(p,q) = 1$, then it is possible to choose $R(p^\beta) \subseteq \Gamma^1(q)$. For $N = \left(\begin{smallmatrix} a & b/2 \\ b/2 & c \end{smallmatrix}\right)$, let $\left(\begin{smallmatrix} a_u & b_u/2 \\ b_u/2 & c_u \end{smallmatrix}\right) = UN^tU$, for a given $U \in SL_2(\mathbb{Z})$.

Theorem 3.1 (Evdokimov [4, (3.7)])**.** *For p prime, with* $\gcd(p,q) = 1$, *and* $N = \left(\begin{smallmatrix} a & b/2 \\ b/2 & c \end{smallmatrix}\right)$,

$$\mathfrak{a}(p^\delta; N) = \sum_{\substack{\alpha+\beta+\gamma=\delta \\ \alpha,\beta,\gamma\geq 0}} \chi_1(p^\beta)\chi_2(p^\gamma)p^{(k-2)\beta+(2k-3)\gamma} \sum_{\substack{U\in R(p^\beta)\subseteq\Gamma^1(q) \\ a_u\equiv 0 \ (\mathrm{mod}\ p^{\beta+\gamma}) \\ b_u\equiv c_u\equiv 0 \ (\mathrm{mod}\ p^\gamma)}} \mathfrak{a}\left(p^\alpha \begin{pmatrix} a_u p^{-\beta-\gamma} & \frac{b_u}{2}p^{-\gamma} \\ \frac{b_u}{2}p^{-\gamma} & c_u p^{\beta-\gamma} \end{pmatrix}\right).$$

4. Igusa Theta Constants and Statement of Results

The Igusa theta constant [6, 7] of degree g with characteristic $m = (m', m'') \in \mathbb{R}^{1\times 2g}$, $m', m'' \in \mathbb{R}^{1\times g}$ is defined, for $Z \in \mathbb{H}^g$, by

$$\theta_m(Z) = \sum_{n\in\mathbb{Z}^{1\times g}} \exp\left(\pi i\left\{(n+m')Z\,{}^t(n+m') + 2(n+m')\,{}^tm''\right\}\right).$$

The product of h theta constants with characteristics $m_1, m_2, \cdots, m_h$ can be written [3]

$$\Theta(Z, M) := \prod_{i=1}^{h} \theta_{m_i}(Z) = \sum_{N\in\mathbb{Z}^{h\times g}} \mathbf{e}\left\{Z\,{}^t(N+M')(N+M') + 2\,{}^tM''(N+M')\right\}$$

where $M = (M', M'')$ with

$$M' = \begin{pmatrix} m'_1 \\ m'_2, \\ \vdots \\ m'_h \end{pmatrix} \quad \text{and} \quad M'' = \begin{pmatrix} m''_1 \\ m''_2, \\ \vdots \\ m''_h \end{pmatrix}.$$

We let $m'_i = (m'_{i1}, m'_{i2}, \ldots, m'_{ig})$ and $m''_i = (m''_{i1}, m''_{i2}, \ldots, m''_{ig})$.

For the remainder of the paper we will assume $h = 2k$ **is even and** $M \in \frac{1}{2}\mathbb{Z}^{h\times 2g}$. Then, $\Theta(Z, M)$ is a modular form of weight k on $\Gamma^g(4,8)$, where

$$\Gamma^g(8) \subset \Gamma^g(4,8) := \left\{\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma^g(4) \mid \mathrm{diag}(B) \equiv \mathrm{diag}(C) \equiv 0 \ (\mathrm{mod}\ 8)\right\} \subset \Gamma^g(4),$$

with Fourier expansion $\Theta(Z, M) = \sum_{N\in\mathcal{R}^g} \mathfrak{a}(N) \exp\left(\frac{2\pi i}{8}\mathrm{Tr}(NZ)\right)$. In fact [3, Theorem 2.2], $\Theta(Z, M) \in \mathfrak{M}_k(\Gamma^g(2), \chi_M)$, where, for $\mathcal{M} = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma^g(2)$,

$$\chi_M(\mathcal{M}) = \rho(\mathrm{Det}(D))^k\, \mathbf{e}\{S(\mathcal{M})\,{}^tMM\},$$

with

$$S(\mathcal{M}) = \begin{pmatrix} B + {}^tB - A\,{}^tB & {}^tD - A\,{}^tD \\ D - I - C\,{}^tB & -C\,{}^tD \end{pmatrix},$$

which is symmetric, and ρ is the non-trivial Dirichlet character modulo four. In particular, if $\mathcal{M} = \left(\begin{smallmatrix} A & 0 \\ 0 & D \end{smallmatrix}\right)$, then $A = {}^tD^{-1}$ and

$$\chi_M(\mathcal{M}) = \rho(\mathrm{Det}(D))^k \exp\left(2\pi i\,\mathrm{Tr}\left((D-I)\,{}^tM'M''\right)\right), \tag{4.1}$$

where we have used the fact that $\mathrm{Tr}(XY) = \mathrm{Tr}(YX)$ when XY is square. We will use $\mathcal{M}$ in this form often. So, for convenience, we define, for $\left(\begin{smallmatrix} {}^tD^{-1} & 0 \\ 0 & D \end{smallmatrix}\right) \in \Gamma^g(2)$,

$$\psi_{_M}(D) := \exp\left(2\pi i \,\mathrm{Tr}\left((D-I)\,{}^tM'M''\right)\right) = \begin{cases} +1 & \text{if } \mathrm{Tr}\left((D-I)\,{}^tM'M''\right) \in \mathbb{Z}, \\ -1 & \text{otherwise,} \end{cases}$$

as $D - I \in 2\mathbb{Z}^{g\times g}$ and ${}^tM'M'' \in \frac{1}{4}\mathbb{Z}^{g\times g}$, necessarily. So, (4.1) becomes

$$\chi_{_M}(\mathcal{M}) = \rho(\mathrm{Det}(D))^k \cdot \psi_{_M}(D). \tag{4.2}$$

Furthermore, if ${}^tM'M'' \in \frac{1}{2}\mathbb{Z}^{g\times g}$, then $\psi_{_M}(D) = 1$ and $\chi_{_M}(\mathcal{M}) = \rho(\mathrm{Det}(D))^k$.

Lemma 4.1. *Let* $\Theta(Z, M) = \sum_{N\in\mathcal{R}^g} \mathfrak{a}(N) \exp\left(\frac{2\pi i}{8}\mathrm{Tr}(NZ)\right)$. *Then*

$$\mathfrak{a}(UN{}^tU) = \mathfrak{a}(N) \cdot \psi_{_M}(U)$$

for all $U \in \mathbb{Z}^{g\times g}$ *with* $\mathrm{Det}\, U = \pm 1$ *and* $U \equiv I \pmod 2$*. In particular, if* ${}^tM'M'' \in \frac{1}{2}\mathbb{Z}^{g\times g}$*, then* $\mathfrak{a}(UN{}^tU) = \mathfrak{a}(N)$*.*

Proof. The result follows from the fact that $\Theta(Z, M) \in \mathfrak{M}_k(\Gamma^g(2), \chi_{_M})$, and applying (3.1) and (4.2). □

Lemma 4.2. *If* $\Theta(Z, M)$ *is not identically zero, then* $\mathrm{Tr}({}^tM'M'') \in \frac{1}{2}\mathbb{Z}$*.*

Proof. From [7, Theorem 2], if $\Theta(Z, M)$ is not identically zero then $m_i'\,{}^tm_i'' \in \frac{1}{2}\mathbb{Z}$ for all $1 \le i \le h$. The result follows as

$$\mathrm{Tr}({}^tM'M'') = \mathrm{Tr}({}^tM''M') = \mathrm{Tr}(M'\,{}^tM'') = \sum_{i=1}^{h} m_i'\,{}^tm_i'',$$

using the facts that $\mathrm{Tr}({}^tX) = \mathrm{Tr}(X)$, and $\mathrm{Tr}(XY) = \mathrm{Tr}(YX)$ when XY is square. □

For the remainder of the paper we will assume $\Theta(Z, M)$ is not identically zero.

Lemma 4.3. *Let* $g = 2$*. Then* $\Theta(Z, M) \in \mathfrak{M}_k(\Gamma^2(8); \rho^{k+\omega}, \varepsilon)$*, where* ε *is trivial and*

$$\omega := \begin{cases} 0 & \text{if } \displaystyle\sum_{i=1}^{h} m_{i1}'m_{i1}'' \in \frac{1}{2}\mathbb{Z}, \\ 1 & \text{otherwise.} \end{cases}$$

In particular, if ${}^tM'M'' \in \frac{1}{2}\mathbb{Z}^{2\times 2}$ *then* $\Theta(Z, M) \in \mathfrak{M}_k(\Gamma^2(8); \rho^k, \varepsilon)$*.*

Proof. Recall, $\Theta(Z, M) \in \mathfrak{M}_k(\Gamma^2(8); \chi_{_1}, \chi_{_2})$ if $\Theta(Z, M)|_k\mathcal{M}_i(m) = \chi_i(m)\Theta(Z, M)$ for all $i \in \{1, 2\}, \gcd(m, 8) = 1$, where $\Gamma^2 \ni \mathcal{M}_i(m) \equiv \left(\begin{smallmatrix} {}^tV_i(m)^{-1} & 0 \\ 0 & V_i(m) \end{smallmatrix}\right) \pmod 8$ and $V_i(m) \in \mathbb{Z}^{2\times 2}$ is the diagonal matrix whose first i diagonal entries equal m and whose remaining diagonal entries equal 1. Thus $\mathcal{M}_i(m) \in \Gamma^2(2)$ and so $\chi_i(m) = \chi_{_M}(\mathcal{M}_i(m))$. By definition, we see that $\chi_{_M}(\mathcal{M})$ is completely determined by $\mathcal{M} \pmod 8$. Therefore,

$$\chi_i(m) = \rho(\mathrm{Det}(V_i(m)))^k \cdot \psi_{_M}(V_i(m)).$$

When $i = 2$, $V_2(m) = \left(\begin{smallmatrix} m & 0 \\ 0 & m \end{smallmatrix}\right)$ and, so, $\mathrm{Tr}\left((V_2(m) - I)\,{}^tM'M''\right) = (m-1)\,\mathrm{Tr}({}^tM'M'') \in \mathbb{Z}$ and $\mathrm{Det}(V_2(m)) = m^2 \equiv 1 \pmod 8$. So, $\chi_2(m) = \varepsilon$. When $i = 1$, $V_1(m) = \left(\begin{smallmatrix} m & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and

$\mathrm{Tr}\left((V_1(m)-I)\,{}^tM'M''\right)=(m-1)\sum_{i=1}^h m'_{i1}m''_{i1}$. Thus, $\psi_{_M}(V_1(m))=1$ if $\sum_{i=1}^h m'_{i1}m''_{i1}\in\frac{1}{2}\mathbb{Z}$, and equals $\rho(m)$ if not. □

Our main results exhibit simple relations between the eigenvalues and the Fourier coefficients of $\Theta(Z,M)$, and also among certain Fourier coefficients, when $\Theta(Z,M)$ is a degree two eigenform. We note that such eigenforms do exist. For example, many of weight three are exhibited in [5].

Theorem 4.4. *Let* $\Theta(Z,M)=\sum_{N\in\mathcal{R}^2}\mathfrak{a}(N)\exp\left(\frac{2\pi i}{8}\mathrm{Tr}(NZ)\right)\in\mathfrak{M}_k(\Gamma^2(8);\rho^{k+\omega},\varepsilon)$ *be an eigenform.*

(1) If $\mathfrak{a}\,(I)=0$*, then* $\mathfrak{a}\,(nI)=0$ *for all* $n\in\mathbb{Z}^+$.

(2) $\mathfrak{a}\,(I)\ \mathfrak{a}\,(mnI)=\mathfrak{a}\,(mI)\ \mathfrak{a}\,(nI)$ *when* $\gcd(m,n)=1$.

(3) $\mathfrak{a}\,(I)\ \mathfrak{a}\left(p^{r+1}I\right)=\mathfrak{a}\,(pI)\ \mathfrak{a}\,(p^rI)-p^{2k-3}\,\mathfrak{a}\,(I)\ \mathfrak{a}\left(p^{r-1}I\right)$

$$-\rho^{k+\omega}(p)\,p^{k-2}\,\mathfrak{a}\,(I)\left[\mathfrak{a}\begin{pmatrix}p^{r+1}&0\\0&p^{r-1}\end{pmatrix}+\mathfrak{a}\begin{pmatrix}p^{r-1}&0\\0&p^{r+1}\end{pmatrix}+\sum_{\substack{u=1\\(8u)^2\not\equiv-1\ (\mathrm{mod}\ p)}}^{p-1}\mathfrak{a}\left(p^{r-1}\begin{pmatrix}1+(8u)^2&8up\\8up&p^2\end{pmatrix}\right)\right],$$

for all odd primes p *and* $r\geq 1$.

Theorem 4.5. *Let* $\Theta(Z,M)=\sum_{N\in\mathcal{R}^2}\mathfrak{a}(N)\exp\left(\frac{2\pi i}{8}\mathrm{Tr}(NZ)\right)\in\mathfrak{M}_k(\Gamma^2(8))$ *be an eigenform, normalized with* $\mathfrak{a}(I)=1$*. Let* $s:=4\,\mathrm{ATr}({}^tM'M'')$*. Then, for any odd prime* p*, the eigenvalues of index* p *and* p^2 *associated to* Θ *satisfy*

$$\lambda(p)=\mathfrak{a}(pI)+\left(1+\left(\tfrac{-1}{p}\right)\right)\cdot\left(\tfrac{2}{p}\right)^s\cdot p^{k-2}$$

and

$$\lambda(p^2)=\mathfrak{a}(p^2I)+\left(1+\left(\tfrac{-1}{p}\right)\right)\cdot\left(\left(\tfrac{2}{p}\right)^s\cdot p^{k-2}\cdot\mathfrak{a}(pI)+p^{2k-4}\right).$$

We note that if $\mathfrak{a}(I)=0$ then it will not be possible to normalize $\mathfrak{a}(I)=1$. However, in this case, it should be possible to produce a similar result to Theorem 4.5, albeit more complicated, leveraged from a non-zero coefficient, using similar methods. We note that eigenforms constructed from theta constants with $\mathfrak{a}(I)\neq 0$ do exist. In fact, eight of the 11 eigenforms listed in [5] have this property.

5. Proofs

We will first need the following lemma which examines the sets $R(p^\beta)\subseteq\Gamma^1(q)$, for $\beta=0,1,2$, which appear in Theorem 3.1.

Lemma 5.1. *For a positive integer* q *with* $\gcd(p,q)=1$*, we can choose*

(1) $R(p^0)=\{\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)\}$;

(2) $R(p^1)=\{\left(\begin{smallmatrix}1&uq\\0&1\end{smallmatrix}\right)\mid u=0,1,\cdots,p-1\}\cup\{\left(\begin{smallmatrix}vp&q\\-wq&1\end{smallmatrix}\right)\mid vp+wq^2=1\}$*; and*

(3) $R(p^2)=\{\left(\begin{smallmatrix}1&uq\\0&1\end{smallmatrix}\right)\mid u=0,\cdots,p^2-1\}\cup\{\left(\begin{smallmatrix}v_up&uq\\-w_uq&1\end{smallmatrix}\right)\mid v_up+w_uuq^2=1,\gcd(v_u,p)=1;u=1,\cdots,p-1\}\cup\{\left(\begin{smallmatrix}vp^2&q\\-wq&1\end{smallmatrix}\right)\mid vp^2+wq^2=1\}$,

and all are contained in $\Gamma^1(q)$.

Proof of Lemma 5.1. In [9, Lemma 3.1] we showed that we could choose

$$R(p^0) = \{\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)\};$$

$$R(p^1) = \{\left(\begin{smallmatrix}1&u\\0&1\end{smallmatrix}\right) \mid u = 0, 1, \cdots, p-1\} \cup \{\left(\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right)\}; and$$

$$R(p^2) = \{\left(\begin{smallmatrix}1&u\\0&1\end{smallmatrix}\right) \mid u = 0, 1, \cdots, p^2-1\} \cup \{\left(\begin{smallmatrix}up&1\\-1&0\end{smallmatrix}\right) \mid u = 0, \cdots, p-1\}.$$

So it suffices to prove that these representations are equivalent, via (3.4), to those in the statement of the lemma. Both representations for $R(p^0)$ are the same so (1) follows automatically. (2): As $\gcd(p, q) = 1$,

$$\{uq \mid u = 0, \cdots, p-1\} \overset{(p)}{\equiv} \{u \mid u = 0, \cdots, p-1\}$$

and so

$$\{(1, uq) \mid u = 0, \cdots, p-1\} \sim \{(1, u) \mid u = 0, \cdots, p-1\},$$

in some order. Choose integers v and w such that $vp + wq^2 = 1$. Then

$$(vp, q) \sim (0, q) \sim (0, 1),$$

where in the last relation we have used (3.4) with l equal to the inverse of q modulo p. (3): In this case we will be applying the relation $\sim$ from (3.4) modulo p^2. As $\gcd(p, q) = 1$, we get that

$$\{(1, uqp) \mid u = 0, \cdots, p-1\} \sim \{(1, up) \mid u = 0, \cdots, p-1\},$$

and

$$\{(1, uq) \mid u \in \left(\mathbb{Z}/p^2\mathbb{Z}\right)^{\times}\} \sim \{(1, u) \mid u \in \left(\mathbb{Z}/p^2\mathbb{Z}\right)^{\times}\}.$$

So, combining these we see that

$$\{(1, uq) \mid u = 0, \cdots, p^2-1\} \sim \{(1, u) \mid u = 0, \cdots, p^2-1\}.$$

Let $u \in \{1, \cdots, p-1\}$. Note $\gcd(p, uq^2) = 1$, so we can choose integers v_u and w_u such that $v_u p + w_u u q^2 = 1$ with $\gcd(v_u, p) = 1$. Let u^{-1} be the inverse of u modulo p and let $u' \in \{1, \cdots, p-1\}$ be the unique integer satisfying $u'q \equiv v_u \pmod{p}$. Then

$$(up, 1) \sim (pq, u^{-1}q) \sim (v_u p, u'u^{-1}q),$$

and, consequently,

$$\{(up, 1) \mid u = 1, \cdots, p-1\} \sim \{(v_u p, uq) \mid u = 1, \cdots, p-1\}$$

in some order. Finally, we choose integers v and w such that $vp^2 + wq^2 = 1$ and note

$$(0, 1) \sim (0, q) \sim (vp^2, q).$$

□

We will also need the following result.

Lemma 5.2. *Let $p \equiv 1 \pmod{4}$ be prime and let β be a positive integer. Then, for an even integer μ satisfying $\mu^2 \equiv -1 \pmod{p^\beta}$, there exists $S \in \Gamma(2)$ such that*

$$S\,{}^tS = \begin{pmatrix} (1+\mu^2)/p^\beta & \mu \\ \mu & p^\beta \end{pmatrix}.$$

Proof. From [9, Lemma 3.3] we know that for any μ, not necessarily even, satisfying $\mu^2 \equiv -1 \pmod{p^\beta}$, there exists $S \in SL_2(\mathbb{Z})$ with the desired property. Furthermore, we know that S is one of

$$S_1 = \begin{pmatrix} (\mu y_\beta + x_\beta)/p^\beta & (\mu x_\beta - y_\beta)/p^\beta \\ y_\beta & x_\beta \end{pmatrix} \text{ or } S_2 = \begin{pmatrix} (\mu y_\beta - x_\beta)/p^\beta & -(\mu x_\beta + y_\beta)/p^\beta \\ y_\beta & -x_\beta \end{pmatrix},$$

depending on which one is integral, where x_β, y_β are integers such that $p^\beta = x_\beta^2 + y_\beta^2$ with x_β odd, y_β even and $p \nmid x_\beta, p \nmid y_\beta$. If μ is even then it easy to see that $S \equiv I \pmod 2$. ☐

To prove our main results we will need to simplify and evaluate the result in Theorem 3.1 under certain circumstances. Corollaries (5.3) - (5.5) are the results of these efforts. We adopt the usual convention that $\mathfrak{a}(N) = 0$ if $N \notin \mathcal{R}^2$.

Corollary 5.3. *For p prime, with* $\gcd(p, q) = 1$, *and* $N = \left(\begin{smallmatrix} a & b/2 \\ b/2 & c \end{smallmatrix}\right)$ *with* $c \not\equiv 0 \pmod p$,

$$\mathfrak{a}(p^\delta; N) = \mathfrak{a}(p^\delta N) + \sum_{\beta=1}^{\delta} \chi_1(p^\beta)\, p^{(k-2)\beta} \sum_{\substack{u=0 \\ a+buq+c(uq)^2 \equiv 0\, (p^\beta)}}^{p^\beta - 1} \mathfrak{a}\left(p^{\delta-\beta} \left(\begin{smallmatrix} (a+buq+c(uq)^2)p^{-\beta} & b/2+cuq \\ b/2+cuq & cp^\beta \end{smallmatrix}\right)\right).$$

Corollary 5.4. *For p prime, with* $\gcd(p, q) = 1$, *and* $N = \left(\begin{smallmatrix} a & b/2 \\ b/2 & c \end{smallmatrix}\right)$,

$$\begin{aligned} \mathfrak{a}(p; N) &= \mathfrak{a}(pN) + \chi_2(p)\, p^{2k-3}\, \mathfrak{a}(p^{-1}N) \\ &\quad + \chi_1(p)\, p^{k-2} \sum_{\substack{u=0 \\ a+buq+c(uq)^2 \equiv 0\, (p)}}^{p-1} \mathfrak{a}\left(\begin{smallmatrix} (a+buq+c(uq)^2)p^{-1} & b/2+cuq \\ b/2+cuq & cp \end{smallmatrix}\right) \\ &\quad + \chi_1(p)\, p^{k-2}\, \mathfrak{a}\left(\begin{smallmatrix} (a(vp)^2+bvpq+cq^2)p^{-1} & -avpwq+b(vp-wq^2)/2+cq \\ -avpwq+b(vp-wq^2)/2+cq & (a(wq)^2-bwq+c)p \end{smallmatrix}\right), \end{aligned}$$

where v and w are any integers satisfying $vp + wq^2 = 1$.

Corollary 5.5. *Let* $F(Z) = \Theta(Z, M) \in \mathfrak{M}_k(\Gamma^2(8); \rho^{k+\omega}, \varepsilon)$. *Then, for p an odd prime and $m \in \mathbb{Z}^+$ such that $(m, p) = 1$,*

$$\mathfrak{a}(p^\delta; mI) = \mathfrak{a}(mp^\delta I) + \begin{cases} 2\displaystyle\sum_{\beta=1}^{\delta} p^{(k-2)\beta}\, \mathfrak{a}(mp^{\delta-\beta} I) & \text{if } p \equiv 1 \pmod 8, \text{or,} \\ & \text{if } p \equiv 5 \pmod 8 \text{ and } \mathrm{ATr}({}^tM'M'') \in \tfrac{1}{2}\mathbb{Z}, \\ 2\displaystyle\sum_{\beta=1}^{\delta} (-1)^\beta\, p^{(k-2)\beta}\, \mathfrak{a}(mp^{\delta-\beta} I) & \text{if } p \equiv 5 \pmod 8 \text{ and } \mathrm{ATr}({}^tM'M'') \notin \tfrac{1}{2}\mathbb{Z}, \\ 0 & \text{if } p \equiv 3 \pmod 4. \end{cases}$$

Proof of Corollary 5.3. Consider Theorem 3.1. Let $U = \left(\begin{smallmatrix} u_1 & u_2 \\ u_3 & u_4 \end{smallmatrix}\right)$. Then

$$\begin{pmatrix} a_u & b_u/2 \\ b_u/2 & c_u \end{pmatrix} = \begin{pmatrix} au_1^2 + bu_1u_2 + cu_2^2 & au_1u_3 + \frac{b}{2}(u_1u_4 + u_2u_3) + cu_2u_4 \\ au_1u_3 + \frac{b}{2}(u_1u_4 + u_2u_3) + cu_2u_4 & au_3^2 + bu_3u_4 + cu_4^2 \end{pmatrix}.$$

We first consider the case when $\beta = 0$. By Lemma 5.1, $R(p^0) = \{I\}$ and so $U = I$ is the only term to consider in the second sum. In which case $c_u = c$. The condition on the second sum that $c_u \equiv 0 \pmod{p^\gamma}$ then implies $\gamma = 0$, as $c \not\equiv 0 \pmod{p}$, and so $\alpha = \delta$. Therefore, the contribution to $\mathfrak{a}(p^\delta; N)$ in the $\beta = 0$ case is $\mathfrak{a}(p^\delta N)$.

Now we consider when $\beta \geq 1$. The condition $a_u \equiv 0 \pmod{p^{\beta+\gamma}}$ implies $p \mid au_1^2 + bu_1u_2 + cu_2^2$. If $p \mid u_1$ then $p \nmid u_2$ as $(u_1, u_2) = 1$ and so $p \mid c$. But $c \not\equiv 0 \pmod{p}$, so $p \nmid u_1$. In this case $(u_1, u_2) \sim (1, u) \pmod{p^\beta}$, where $u \in \{0, 1, \cdots, p^\beta - 1\}$ with $u \equiv u_1^{-1} u_2 \pmod{p^\beta}$, and u_1^{-1} is the inverse of u_1 in $\left(\mathbb{Z}/p^\beta\mathbb{Z}\right)^\times$. So we need only consider U in any subset of $R(p^\beta) \subseteq \Gamma^1(q)$ that is $\sim$ equivalent to $\{\left(\begin{smallmatrix} 1 & u \\ 0 & 1 \end{smallmatrix}\right) \mid u = 0, 1, \cdots, p^\beta - 1\}$. We choose $\{\left(\begin{smallmatrix} 1 & uq \\ 0 & 1 \end{smallmatrix}\right) \mid u = 0, 1, \cdots, p^\beta - 1\}$ for this subset. Note that if $U = \left(\begin{smallmatrix} 1 & uq \\ 0 & 1 \end{smallmatrix}\right)$ then

$$\begin{pmatrix} a_u & \frac{b_u}{2} \\ \frac{b_u}{2} & c_u \end{pmatrix} = \begin{pmatrix} a + buq + c(uq)^2 & \frac{b}{2} + cuq \\ \frac{b}{2} + cuq & c \end{pmatrix}.$$

In particular $c_u = c$ and so the condition that $c_u \equiv 0 \pmod{p^\gamma}$ implies $\gamma = 0$. Thus the contribution to $\mathfrak{a}(p^\delta; N)$ in the case $\beta \geq 1$ is

$$\sum_{\beta=1}^{\delta} \chi_1(p^\beta)\, p^{(k-2)\beta} \sum_{\substack{u=0 \\ a+buq+c(uq)^2 \equiv 0\,(p^\beta)}}^{p^\beta - 1} \mathfrak{a}\left(p^{\delta-\beta} \left(\begin{smallmatrix} (a+buq+c(uq)^2)p^{-\beta} & b/2+cuq \\ b/2+cuq & cp^\beta \end{smallmatrix}\right)\right).$$

□

Proof of Corollary 5.4. This follows easily from taking $\delta = 1$ in Theorem 3.1 and applying Lemma 5.1. □

Proof of Corollary 5.5. Taking $N = mI$ in Corollary 5.3 and applying to $\Theta(Z, M) \in \mathfrak{M}_k(\Gamma^2(8); \rho^{k+\omega}, \varepsilon)$ we get that

$$\mathfrak{a}(p^\delta; mI) = \mathfrak{a}(mp^\delta I) + \sum_{\beta=1}^{\delta} \rho^{k+\omega}(p^\beta)\, p^{(k-2)\beta} \sum_{\substack{u=0 \\ m+m(8u)^2 \equiv 0\,(p^\beta)}}^{p^\beta - 1} \mathfrak{a}\left(mp^{\delta-\beta} \left(\begin{smallmatrix} (1+(8u)^2)p^{-\beta} & 8u \\ 8u & p^\beta \end{smallmatrix}\right)\right)$$

Now $m + m(8u)^2 \equiv 0 \pmod{p^\beta} \Leftrightarrow (8u)^2 \equiv -1 \pmod{p^\beta}$ as $(m, p) = 1$. If $p \equiv 3 \pmod{4}$ there is no such u and so $\mathfrak{a}(p^\delta; mI) = \mathfrak{a}(mp^\delta I)$. Now we examine the case when $p \equiv 1 \pmod{4}$. In this case, $(8u)^2 \equiv -1 \pmod{p^\beta}$ has two distinct solutions. From Lemma 5.2, with $\mu = 8u$, we know there exists $S \in \Gamma(2)$ such that

$$S\, {}^tS = \left(\begin{smallmatrix} (1+(8u)^2)/p^\beta & 8u \\ 8u & p^\beta \end{smallmatrix}\right).$$

Therefore

$$S\, mp^{\delta-\beta} I\, {}^tS = mp^{\delta-\beta} \left(\begin{smallmatrix} (1+(8u)^2)/p^\beta & 8u \\ 8u & p^\beta \end{smallmatrix}\right),$$

and so by Lemma 4.1 we see that

$$\mathfrak{a}\left(mp^{\delta-\beta} \left(\begin{smallmatrix} (1+(8u)^2)/p^\beta & 8u \\ 8u & p^\beta \end{smallmatrix}\right)\right) = \mathfrak{a}(mp^{\delta-\beta} I) \cdot \psi_M(S).$$

Now, $\psi_M(S) = \exp\left(2\pi i \operatorname{Tr}\left((S - I)\, {}^tM' M''\right)\right)$, so we need only consider S modulo 4 as ${}^tM'M'' \in \frac{1}{4}\mathbb{Z}^{g \times g}$, necessarily. From the proof of Lemma 5.2, we see that $S = S_1 \equiv$

$\left(\begin{smallmatrix} x_\beta & -y_\beta \\ y_\beta & x_\beta \end{smallmatrix}\right) \pmod 4$ or $S = S_2 \equiv \left(\begin{smallmatrix} -x_\beta & -y_\beta \\ y_\beta & -x_\beta \end{smallmatrix}\right) \pmod 4$, where x_β, y_β are integers such that $p^\beta = x_\beta^2 + y_\beta^2$ with x_β odd, y_β even and $p \nmid x_\beta, p \nmid y_\beta$. So,

$$\begin{aligned} \psi_{_M}(S) &= \exp\left(2\pi i \ \left((\pm x_\beta - 1) \cdot \mathrm{Tr}({}^tM'M'') + y_\beta \cdot \mathrm{ATr}({}^tM'M'')\right)\right) \\ &= \exp\left(2\pi i \ \left(y_\beta \cdot \mathrm{ATr}({}^tM'M'')\right)\right) \end{aligned}$$

as $\pm x_\beta - 1$ is even and $\mathrm{Tr}({}^tM'M'') \in \frac{1}{2}\mathbb{Z}$ by Lemma 4.2. If $p^\beta \equiv 1 \pmod 8$ then $y_\beta \equiv 0 \pmod 4$ and $\psi_{_M}(S) = 1$. If $\mathrm{ATr}({}^tM'M'') \in \frac{1}{2}\mathbb{Z}$ then $\psi_{_M}(S) = 1$. The only other possibility is that $p^\beta \equiv 5 \pmod 8$, i.e., $p \equiv 5 \pmod 8$ with β odd, and $\mathrm{ATr}({}^tM'M'') \notin \frac{1}{2}\mathbb{Z}$. In this case $y_\beta \equiv 2 \pmod 4$ and $\psi_{_M}(S) = -1$. □

Proof of Theorem 4.5. We take $m = 1$ in Corollary 5.5 to get that

$$\mathfrak{a}(p^\delta; I) = \mathfrak{a}(p^\delta I) + \begin{cases} 2\displaystyle\sum_{\beta=1}^{\delta} p^{(k-2)\beta}\,\mathfrak{a}(p^{\delta-\beta}I) & \text{if } p \equiv 1 \pmod 8, \text{or}, \\ & \text{if } p \equiv 5 \pmod 8 \text{ and } \mathrm{ATr}({}^tM'M'') \in \frac{1}{2}\mathbb{Z}, \\ 2\displaystyle\sum_{\beta=1}^{\delta} (-1)^\beta p^{(k-2)\beta}\,\mathfrak{a}(p^{\delta-\beta}I) & \text{if } p \equiv 5 \pmod 8 \text{ and } \mathrm{ATr}({}^tM'M'') \notin \frac{1}{2}\mathbb{Z}, \\ 0 & \text{if } p \equiv 3 \pmod 4. \end{cases}$$

Taking $\delta = 1, 2$ and then applying (3.3), noting that $\mathrm{ATr}({}^tM'M'') \in \frac{1}{4}\mathbb{Z}$, $\mathfrak{a}(I) = 1$ and

$$\left(1 + \left(\tfrac{-1}{p}\right)\right) \cdot \left(\tfrac{2}{p}\right)^s = \begin{cases} 2 & \text{if } p \equiv 1 \pmod 8, \text{or}, \ p \equiv 5 \pmod 8 \text{ and } \mathrm{ATr}({}^tM'M'') \in \frac{1}{2}\mathbb{Z}, \\ -2 & \text{if } p \equiv 5 \pmod 8 \text{ and } \mathrm{ATr}({}^tM'M'') \notin \frac{1}{2}\mathbb{Z}, \\ 0 & \text{if } p \equiv 3 \pmod 4, \end{cases}$$

yields the result. □

Proof of Theorem 4.4. Let $s := 4\,\mathrm{ATr}({}^tM'M'')$. (1) We take $\delta = 1$ in Corollary 5.5 and then apply (3.3) to get that, for $(m, p) = 1$,

$$\mathfrak{a}(mI)\lambda(p) = \mathfrak{a}(mpI) + \left(1 + \left(\tfrac{-1}{p}\right)\right) \cdot \left(\tfrac{2}{p}\right)^s \cdot p^{k-2}\,\mathfrak{a}(mI). \tag{5.1}$$

Therefore, if $a(mI) = 0$ then $a(mpI) = 0$, whenever $(m, p) = 1$. Inductively, we can then show, using Corollary 5.5, that

$$a(mI) = 0 \Rightarrow a(mp^\delta I) = 0, \tag{5.2}$$

for any $\delta \in \mathbb{Z}^+$, whenever $(m, p) = 1$. If $n = p_1^{\delta_1} p_2^{\delta_2} \cdots p_t^{\delta_t}$ for distinct primes $p_1, p_2, \cdots p_t$, then repeated use of (5.2) yields

$$a(I) = 0 \Rightarrow a(p_1^{\delta_1} I) = 0 \Rightarrow a(p_1^{\delta_1} p_2^{\delta_2} I) = 0 \Rightarrow \cdots \Rightarrow a(nI) = 0,$$

as required.

(2) It suffices to prove

$$\mathfrak{a}(I)\mathfrak{a}(mp^\delta I) = \mathfrak{a}(mI)\mathfrak{a}(p^\delta I)$$

for all $\delta \in \mathbb{Z}^+$ and $(m,p)=1$. We prove this by induction on δ. Taking $m=1$ in (5.1) we have

$$\mathfrak{a}(I)\lambda(p) = \mathfrak{a}(pI) + \left(1+\left(\tfrac{-1}{p}\right)\right)\cdot\left(\tfrac{2}{p}\right)^s \cdot p^{k-2}\,\mathfrak{a}(I). \tag{5.3}$$

Then $\mathfrak{a}(I)$ times (5.1) minus $\mathfrak{a}(mI)$ times (5.3) tells us that

$$\mathfrak{a}(I)\mathfrak{a}(mpI) = \mathfrak{a}(mI)\mathfrak{a}(pI) \tag{5.4}$$

when $(m,p)=1$. In a similar manner, but this time using Corollary 5.5, with δ unrestricted, instead of (5.1), we get that

$$\mathfrak{a}(I)\mathfrak{a}(mp^\delta I) = \mathfrak{a}(mI)\mathfrak{a}(p^\delta I)$$

where we have used the fact that

$$\mathfrak{a}(I)\mathfrak{a}(mp^{\delta-\beta} I) = \mathfrak{a}(mI)\mathfrak{a}(p^{\delta-\beta} I)$$

for all $1 \leq \beta \leq \delta$, by the induction hypotheses and (5.4).

(3) Taking $N = p^r I$ in Corollary 5.4 and applying to $\Theta(Z,M) \in \mathfrak{M}_k(\Gamma^2(8);\rho^{k+\omega},\varepsilon)$ we get that

$$\mathfrak{a}(p;p^rI) = \mathfrak{a}(p^{r+1}I) + p^{2k-3}\,\mathfrak{a}(p^{r-1}I) + \rho^{k+\omega}(p)\,p^{k-2}\,\mathfrak{a}\left(p^{r-1}\left(\begin{smallmatrix} q^2+(pv)^2 & pq-p^2vwq \\ pq-p^2vwq & p^2(1+(wq)^2)\end{smallmatrix}\right)\right)$$
$$+\,\rho^{k+\omega}(p)\,p^{k-2}\sum_{u=0}^{p-1}\mathfrak{a}\left(p^{r-1}\left(\begin{smallmatrix} (1+(uq)^2 & puq \\ puq & p^2\end{smallmatrix}\right)\right), \tag{5.5}$$

where $q=8$, p is odd, and, v and w are any integers satisfying $vp+wq^2=1$. Applying Lemma 4.1 with $U = \left(\begin{smallmatrix} v & q \\ -wq & p\end{smallmatrix}\right)$ and $N = \left(\begin{smallmatrix} p^{r+1} & 0 \\ 0 & p^{r-1}\end{smallmatrix}\right)$ we have that

$$\mathfrak{a}\left(p^{r-1}\left(\begin{smallmatrix} q^2+(pv)^2 & pq-p^2vwq \\ pq-p^2vwq & p^2(1+(wq)^2)\end{smallmatrix}\right)\right) = \mathfrak{a}\left(\begin{smallmatrix} p^{r+1} & 0 \\ 0 & p^{r-1}\end{smallmatrix}\right)\cdot\psi_{_M}(U). \tag{5.6}$$

By definition, to evaluate $\psi_{_M}(U)$, we need only consider U modulo 4, as ${}^tM'M'' \in \frac{1}{4}\mathbb{Z}^{g\times g}$. Note $q=8$ and $v \equiv p \pmod 4$. Therefore,

$$\psi_{_M}(U) = \exp\left(2\pi i\,\left((p-1)\cdot\mathrm{Tr}({}^tM'M'')\right)\right) = 1,$$

as p odd and $\mathrm{Tr}({}^tM'M'') \in \frac{1}{2}\mathbb{Z}^{g\times g}$ by Lemma 4.2.

We now consider the sum in (5.5) when $(uq)^2 \equiv -1 \pmod p$, which has exactly two solutions in u when $p \equiv 1 \pmod 4$, and none when $p \equiv 3 \pmod 4$. So, when $p \equiv 1 \pmod 4$, by Lemma 5.2, with $\mu = uq$ and $\beta = 1$, there exists $S \in \Gamma(2)$ such that

$$S\,{}^tS = \left(\begin{smallmatrix} (1+(uq)^2)/p & uq \\ uq & p\end{smallmatrix}\right).$$

Then applying Lemma 4.1 with $U=S$ and $N=p^rI$ we get

$$\mathfrak{a}\left(p^{r-1}\left(\begin{smallmatrix} (1+(uq)^2 & puq \\ puq & p^2\end{smallmatrix}\right)\right) = \mathfrak{a}(p^rI)\cdot\psi_{_M}(S) \tag{5.7}$$

From the proof of Corollary 5.5 we see that, when $\beta=1$,

$$\psi_{_M}(S) = \begin{cases} 1 & \text{if } p\equiv 1 \pmod 8, \text{or, } p \equiv 5 \pmod 8 \text{ and } \mathrm{ATr}({}^tM'M'') \in \frac{1}{2}\mathbb{Z}, \\ -1 & \text{if } p \equiv 5 \pmod 8 \text{ and } \mathrm{ATr}({}^tM'M'') \notin \frac{1}{2}\mathbb{Z}. \end{cases}$$

So $\psi_{_M}(S) = \left(\frac{2}{p}\right)^s$. Now, substituting for (5.6) and (5.7) in (5.5) and splitting off the $u=0$ term yields

$$\mathfrak{a}(p;p^rI) = \mathfrak{a}(p^{r+1}I) + p^{2k-3}\,\mathfrak{a}(p^{r-1}I) + \rho^{k+\omega}(p)\,p^{k-2}\left[\mathfrak{a}\begin{pmatrix} p^{r+1} & 0 \\ 0 & p^{r-1}\end{pmatrix} + \mathfrak{a}\begin{pmatrix} p^{r-1} & 0 \\ 0 & p^{r+1}\end{pmatrix}\right.$$
$$\left. + \left(1+\left(\tfrac{-1}{p}\right)\right)\cdot\left(\tfrac{2}{p}\right)^s\cdot\mathfrak{a}(p^rI) + \sum_{\substack{u=1 \\ (uq)^2\not\equiv -1 \pmod p}}^{p-1} \mathfrak{a}\left(p^{r-1}\begin{pmatrix} 1+(uq)^2 & puq \\ puq & p^2\end{pmatrix}\right)\right]. \quad (5.8)$$

Now $\mathfrak{a}(I)\mathfrak{a}(p;p^rI) = \mathfrak{a}(I)\mathfrak{a}(p^rI)\lambda(p) = \mathfrak{a}(p^rI)\mathfrak{a}(p;I)$. Accounting for (5.8) and (5.3) in the previous statement yields the result. □

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Data Availability

Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

Compliance with Ethical Standards

The author certifies that he has no conflicts of interest.

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