ON A QUESTION OF ZHANG

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Abstract. When studying the conjugacy class version of the Huppert's ρ-σ conjecture, Jiping Zhang raised a number theory question. In this paper, we provide examples to show that the question raised by Zhang is not always true in general.

1. Introduction

When studying the conjugacy class version of the Huppert's ρ-σ conjecture, Jiping Zhang raised a number theory conjecture [2, Problem in p.2399]. Zhang claimed that if the statement would be true, then he could use it to show the best possible bound for the conjugacy class version of the Huppert's ρ-σ conjecture. We believe that the techniques are about the detailed analysis of the orbit structure of the solvable linear groups, and in particular, the orbit structure of semi-linear groups. The problem was asked again in a survey paper of Moretó [1]. In this paper, we provide examples to show that unfortunately the question raised by Zhang is not true in general.

2. Question and counterexamples

Let $k = p_1^{a_1} \cdots p_t^{a_t}$ be a positive integer written as a product of powers of pairwise different primes. We define $\omega(k) = t$.

We first state the question (see [2, Problem in p.2399] and also [1, Question 3.3]). Let $m_1 \ldots, m_n > 1$ be pairwise coprime positive integers and $q_1, \ldots, q_n$ be n arbitrary prime powers. Is it true that

$$\omega\Big(\prod_{i=1}^{n} \frac{q_i^{m_i} - 1}{q_i - 1}\Big) \geq n?$$

We provide the following counterexamples to this question.

Let $n = 2$, $q_1 = 2$; $q_2 = 5$ and $m_1 = 5$; $m_2 = 3$. Then

$$\prod_{i=1}^{2} \frac{q_i^{m_i} - 1}{q_i - 1} = \frac{2^5 - 1}{2 - 1} \cdot \frac{5^3 - 1}{5 - 1} = 31^2.$$

So

$$1 = \omega\Big(\prod_{i=1}^{2} \frac{q_i^{m_i} - 1}{q_i - 1}\Big) < n = 2.$$

Similarly, let $n = 3$, $q_1 = 2$; $q_2 = 5$; $q_3 = 3$ and $m_1 = 5$; $m_2 = 3$; $m_3 = 2$. Then

$$\prod_{i=1}^{3} \frac{q_i^{m_i} - 1}{q_i - 1} = \frac{2^5 - 1}{2 - 1} \cdot \frac{5^3 - 1}{5 - 1} \cdot \frac{3^2 - 1}{3 - 1} = 31^2 \cdot 2^2.$$

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So

$$2 = \omega\Big(\prod_{i=1}^{3} \frac{q_i^{m_i}-1}{q_i-1}\Big) < n = 3.$$

The following is another one. Let $n = 4$, $q_1 = 2$; $q_2 = 5$; $q_3 = 3$; $q_4 = 3$ and $m_1 = 5$; $m_2 = 3$; $m_3 = 2$; $m_4 = 13$. Then

$$\prod_{i=1}^{4} \frac{q_i^{m_i}-1}{q_i-1} = \frac{2^5-1}{2-1}\cdot\frac{5^3-1}{5-1}\cdot\frac{3^2-1}{3-1}\cdot\frac{3^{13}-1}{3-1} = 31^2\cdot 2^2\cdot 797161.$$

So

$$3 = \omega\Big(\prod_{i=1}^{3} \frac{q_i^{m_i}-1}{q_i-1}\Big) < n = 4.$$

Indeed, we may multiply more suitable Mersenne primes

$$p_j = \frac{2^{m_j}-1}{2-1}$$

in the end of the previous examples to construct larger counterexamples.

In view of those examples, maybe it is reasonable to ask:
Is it true that

$$\omega\Big(\prod_{i=1}^{n} \frac{q_i^{m_i}-1}{q_i-1}\Big) \geq n-k$$

where k is a fixed constant not depending on n?

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References

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