ORBITS OF FINITE FIELD HYPERGEOMETRIC FUNCTIONS AND COMPLETE SUBGRAPHS OF GENERALIZED PALEY GRAPHS

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ABSTRACT. In a recent result of Dawsey and McCarthy, a formula for the number of complete subgraphs of order four of generalized Paley graphs is given in terms of a sum of finite field hypergeometric functions. Via known transformation formulas for finite field hypergeometric functions, many of the summands in their formula are equal. They construct a group action representing these transformations so that the number of summands that need to be evaluated is reduced to orbit representatives. In this paper, we expand the group used by Dawsey and McCarthy, reducing by up to 80% the number of summands to be evaluated.

1. INTRODUCTION

Finite field hypergeometric functions, introduced by Greene [9, 10] as analogues of classical hypergeometric series, have nice character sum representations and, consequently, lend themselves naturally to counting problems (see, for example, [1, 2, 8, 11, 13]). Recently, Dawsey and McCarthy [6] provide a formula for the number of complete subgraphs of order four of generalized k-th power Paley graphs in terms of a sum of ${}_3F_2$ finite field hypergeometric functions. Using this result, they give lower bounds for multicolor diagonal Ramsey numbers.

Let $\mathbb{F}_q$ denote the finite field with q, a prime power, elements. For a multiplicative character χ of $\mathbb{F}_q^*$, we extend its domain to $\mathbb{F}_q$ by defining $\chi(0) := 0$ (including for the trivial character). For characters A, B, C, D, E of $\mathbb{F}_q^*$ and $\lambda \in \mathbb{F}_q^*$, we define the ${}_3F_2$ finite field hypergeometric function [10, Cor. 3.14 (after change of variable)] by

$$ {}_3F_2\left(\begin{array}{ccc} A, & B, & C \\ & D, & E \end{array} \Big| \, \lambda \right)_q = \frac{1}{q^2} \sum_{a,b \in \mathbb{F}_q} A\overline{E}(a)\overline{C}E(1-a)B(b)\overline{B}D(b-1)\overline{A}(a-\lambda b). $$

Let $k \geq 2$ be an integer. Let q be such that $q \equiv 1 \pmod{k}$ if q is even, or, $q \equiv 1 \pmod{2k}$ if q is odd. These conditions guarantee that -1 is a k-th power in $\mathbb{F}_q$. The *generalized Paley graph* of order q, $G_k(q)$, is the graph with vertex set $\mathbb{F}_q$ where ab is an edge if and only if $a - b$ is a k-th power residue [5, 12]. Let χ_k be a character of $\mathbb{F}_q^*$ of order k. For $\vec{t} = (t_1, t_2, t_3, t_4, t_5) \in (\mathbb{Z}_k)^5$, we define, for reasons of brevity,

$$ {}_3F_2\left(\vec{t} \,\middle|\, \lambda\right)_{q,k} := {}_3F_2\left(\begin{array}{ccc} \chi_k^{t_1}, & \chi_k^{t_2}, & \chi_k^{t_3} \\ & \chi_k^{t_4}, & \chi_k^{t_5} \end{array} \Big| \, \lambda \right)_q. $$

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Then, the number of complete subgraphs of order four contained in $G_k(q)$, $\mathcal{K}_4(G_k(q))$, is given by [6, Thm 2.1]

$$\mathcal{K}_4(G_k(q)) = \frac{q^3(q-1)}{24 \cdot k^6} \sum_{\vec{t} \in (\mathbb{Z}_k)^5} {}_3F_2\left(\vec{t} \mid 1\right)_{q,k}. \tag{1.1}$$

Evaluating $\mathcal{K}_4(G_k(q))$, using (1.1), we can find lower bounds for the multicolor diagonal Ramsey numbers $R_k(4) = R(4, 4, \cdots, 4)$. Specifically, for a given k, if $\mathcal{K}_4(G_k(q)) = 0$ for some q, then $q < R_k(4)$. Key to evaluating $\mathcal{K}_4(G_k(q))$ is being able to simplify and evaluate the hypergeometric functions in (1.1).

Certain summands in (1.1) can be simplified using known reduction formulae for finite field hypergeometric functions. After splitting off these terms [6, Thm 2.2], the remaining hypergeometric terms are

$$\sum_{\vec{t} \in X_k} {}_3F_2\left(\vec{t} \mid 1\right)_{q,k}. \tag{1.2}$$

where $X_k := \{(t_1, t_2, t_3, t_4, t_5) \in (\mathbb{Z}_k)^5 \mid t_1, t_2, t_3 \neq 0, t_4, t_5\,;\, t_1 + t_2 + t_3 \neq t_4 + t_5\}$.

Many of the hypergeometric summands that still remain in (1.2) are equal. A group action on X_k, such that the value of ${}_3F_2\left(\vec{t} \mid 1\right)_{q,k}$ is invariant in each orbit, is described in [6]. Consequently, the number of hypergeometric terms that need to be evaluated is reduced to orbit representatives. This technique and action has since been used by Bhowmik and Barman in relation to Peisert [3] and Peisert-like [4] graphs.

In this paper, we show that a larger group acts on X_k with the same effects and so the number of orbits, and, hence, the number of hypergeometric terms that need to be evaluated, can be reduced even further. Table 1 below outlines the level of this reduction. Let N_k be the number of orbits for a given k.

k	$\lvert X_k \rvert$	N_k from [6]	N_k from this paper	Reduction
2	1	1	1	0%
3	12	1	1	0%
4	93	11	6	45%
5	424	28	12	57%
6	1425	92	33	64%
7	3876	207	63	70%
8	9037	466	131	72%
∞				80%

Table 1. Reduction in Number of Orbits

So, as a result of this work, as k becomes large, the number of hypergeometric terms in (1.2) that need to be evaluated will be automatically up to 80% less.

In the next section, we describe in detail the group action and give a formula for the number of orbits. In the appendix, we give Python code which returns the orbits for a given k.

2. Orbits of X_k

An important feature of finite field hypergeometric functions is transformation formulae relating the values of functions with different parameters (analogous to those for classical hypergeometric series). Seven such transformations, applicable to ${}_3F_2(\cdot|1)$, are identified in [6, (3.15)-(3.21)], having originally appearing in [9, 10]. For example [10, Thm 4.2],

$$ {}_3F_2\left(\begin{matrix} A, & B, & C \\ & D, & E \end{matrix} \;\middle|\; 1\right)_q = {}_3F_2\left(\begin{matrix} B\overline{D}, & A\overline{D}, & C\overline{D} \\ & \overline{D}, & E\overline{D} \end{matrix} \;\middle|\; 1\right)_q . \tag{2.1} $$

To each such transformation we can associate a map on X_k. For example, applying (2.1) we get that

$$ {}_3F_2\left(\begin{matrix} \chi_k^{t_1}, & \chi_k^{t_2}, & \chi_k^{t_3} \\ & \chi_k^{t_4}, & \chi_k^{t_5} \end{matrix} \;\middle|\; 1\right)_q = {}_3F_2\left(\begin{matrix} \chi_k^{t_2-t_4}, & \chi_k^{t_1-t_4}, & \chi_k^{t_3-t_4} \\ & \chi_k^{-t_4}, & \chi_k^{t_5-t_4} \end{matrix} \;\middle|\; 1\right)_q . $$

This induces a map $T_1 : X_k \to X_k$ given by

$$ T_1(t_1, t_2, t_3, t_4, t_5) = (t_2 - t_4, t_1 - t_4, t_3 - t_4, -t_4, t_5 - t_4), $$

where the addition in each component takes place in $\mathbb{Z}_k$. Similarly, to the other transformations [6, (3.16)-(3.21)] we can associate the maps

$$ \begin{aligned} T_2(t_1, t_2, t_3, t_4, t_5) &= (t_1, t_1 - t_4, t_1 - t_5, t_1 - t_2, t_1 - t_3); \\ T_3(t_1, t_2, t_3, t_4, t_5) &= (t_2 - t_4, t_2, t_2 - t_5, t_2 - t_1, t_2 - t_3); \\ T_4(t_1, t_2, t_3, t_4, t_5) &= (t_1, t_2, t_5 - t_3, t_1 + t_2 - t_4, t_5); \\ T_5(t_1, t_2, t_3, t_4, t_5) &= (t_1, t_4 - t_2, t_3, t_4, t_1 + t_3 - t_5); \\ T_6(t_1, t_2, t_3, t_4, t_5) &= (t_4 - t_1, t_2, t_3, t_4, t_2 + t_3 - t_5); \text{ and} \\ T_7(t_1, t_2, t_3, t_4, t_5) &= (t_4 - t_1, t_4 - t_2, t_3, t_4, t_4 + t_5 - t_1 - t_2) \end{aligned} $$

respectively. We form the group generated by $T_1, T_2, \cdots, T_7$, with operation composition of functions, and call it $\mathbb{T}_k$. We have that

$$ \begin{aligned} \mathbb{T}_k &= \langle T_1, T_2, T_3, T_4, T_5, T_6, T_7 \rangle \\ &= \{T_0, T_i, T_j \circ T_\ell, T_4 \circ T_1, T_6 \circ T_2, T_5 \circ T_3, T_1 \circ T_4 \circ T_1 | 1 \le i \le 7, 1 \le j \le 3, 4 \le \ell \le 7\}, \end{aligned} $$

where T_0 is the identity map, is a group of order 24 isomorphic to the permutation group S_4. So, in fact, $\mathbb{T}_k = \langle T_2, T_1 \circ T_6 \rangle$. $\mathbb{T}_k$ acts on X_k. Furthermore, the value of ${}_3F_2\left(\vec{t} \;\middle|\; 1\right)_{q,k}$ is constant for all $\vec{t}$ in each orbit. Therefore, the evaluation of the hypergeometric summands in (1.2) can be reduced to orbit representatives.

$\mathbb{T}_k$ is the group introduced in [6] and the following formula for the number of orbits for a given k, $N_{\mathbb{T}_k}$, is also provided therein, which is calculated using Burnside's theorem.

$$N_{\mathbb{T}_k} = \frac{1}{24}\Bigg[k^5 - 10k^4 + 54k^3 - 162k^2 + 245k - 128 + \begin{cases} 0 & \text{if } k \equiv 1, 5, 7, 11 \pmod{12}, \\ 16k - 64 & \text{if } k \equiv 3, 9 \pmod{12}, \\ 45k - 84 & \text{if } k \equiv 2, 10 \pmod{12}, \\ 45k - 96 & \text{if } k \equiv 4, 8 \pmod{12}, \\ 61k - 148 & \text{if } k \equiv 6 \pmod{12}, \\ 61k - 160 & \text{if } k \equiv 0 \pmod{12}. \end{cases}\Bigg] \tag{2.2}$$

We note also that [6, (6.2)]

$$|X_k| = (k-1)(k^4 - 9k^3 + 36k^2 - 69k + 51). \tag{2.3}$$

It is easy to see from their definition that the value of finite field hypergeometric functions is invariant under permuting columns of parameters, and so

$${}_3F_2\left(\begin{array}{ccc} \chi_k^{t_1}, & \chi_k^{t_2}, & \chi_k^{t_3} \\ & \chi_k^{t_4}, & \chi_k^{t_5} \end{array} \Big| \, 1 \right)_q = {}_3F_2\left(\begin{array}{ccc} \chi_k^{t_1}, & \chi_k^{t_3}, & \chi_k^{t_2} \\ & \chi_k^{t_5}, & \chi_k^{t_4} \end{array} \Big| \, 1 \right)_q,$$

$${}_3F_2\left(\begin{array}{ccc} \chi_k^{t_1}, & \chi_k^{t_2}, & \chi_k^{t_3} \\ & \chi_k^{0}, & \chi_k^{t_5} \end{array} \Big| \, 1 \right)_q = {}_3F_2\left(\begin{array}{ccc} \chi_k^{t_2}, & \chi_k^{t_1}, & \chi_k^{t_3} \\ & \chi_k^{0}, & \chi_k^{t_5} \end{array} \Big| \, 1 \right)_q$$

and

$${}_3F_2\left(\begin{array}{ccc} \chi_k^{t_1}, & \chi_k^{t_2}, & \chi_k^{t_3} \\ & \chi_k^{t_4}, & \chi_k^{0} \end{array} \Big| \, 1 \right)_q = {}_3F_2\left(\begin{array}{ccc} \chi_k^{t_3}, & \chi_k^{t_2}, & \chi_k^{t_1} \\ & \chi_k^{t_4}, & \chi_k^{0} \end{array} \Big| \, 1 \right)_q.$$

These transformations induce the following maps on X_k:

$$\begin{aligned} T_8(t_1, t_2, t_3, t_4, t_5) &= (t_1, t_3, t_2, t_5, t_4); \\ T_9(t_1, t_2, t_3, 0, t_5) &= (t_2, t_1, t_3, 0, t_5); \text{ and} \\ T_{10}(t_1, t_2, t_3, t_4, 0) &= (t_3, t_2, t_1, t_4, 0), \end{aligned}$$

where the domain must be restricted for T_9 and T_{10}. The purpose of this paper is to incorporate these maps in to the group acting on X_k. This larger group reduces the number of orbits, and, consequently, automatically reduces the number of hypergeometric terms in (1.2) that need to be evaluated. We start by adding T_8. Let

$$\begin{aligned} \mathbb{G}_k &= \langle \mathbb{T}_k, T_8 \rangle \\ &= \langle T_1, T_2, T_3, T_4, T_5, T_6, T_7, T_8 \rangle \\ &= \langle T_2, T_1 \circ T_6, T_8 \rangle. \end{aligned}$$

Using a computer, we generate $\mathbb{G}_k$ (see Appendix A) and find that it is a group of order 120 isomorphic to the permutation group S_5. Consider the maps

$$\begin{aligned} G_1(t_1, t_2, t_3, t_4, t_5) &= (t_1, t_2, t_5 - t_3, t_1 + t_2 - t_4, t_5); \text{ and} \\ G_2(t_1, t_2, t_3, t_4, t_5) &= (t_5 - t_1, t_3, t_3 - t_4, t_3 - t_1, t_2 + t_3 - t_4). \end{aligned}$$

Then $G_1 = T_1 \circ T_6 \circ T_1 \circ T_6 \circ T_2 \circ T_1 \circ T_6 \circ T_2$ and $G_2 = T_1 \circ T_6 \circ T_1 \circ T_6 \circ T_1 \circ T_6 \circ T_2 \circ T_8$ are elements of $\mathbb{G}_k$, and, the maps $(12) \to G_1, (12345) \to G_2$ determine an isomorphism with S_5. So, in fact, $\mathbb{G}_k = \langle G_1, G_2 \rangle$ as $S_5 = \langle (12), (12345) \rangle$. As before, $\mathbb{G}_k$ acts on X_k and the value of ${}_3F_2\left(\vec{t} \mid 1\right)_{q,k}$ is constant for all $\vec{t}$ in each orbit. Therefore, we can reduce the evaluation of the hypergeometric summands in (1.2) to $\mathbb{G}_k$ orbit representatives.

We can calculate explicitly the number of orbits in X_k under the action of $\mathbb{G}_k$, for a given k, $N_{\mathbb{G}_k}$. For $T \in \mathbb{G}_k$, let $X_T := \{\vec{t} \in X_k \mid T(\vec{t}\,) = \vec{t}\,\}$. Then, by Burnside's theorem, the number of orbits is given by

$$N_{\mathbb{G}_k} = \frac{1}{120} \sum_{T \in \mathbb{G}_k} |X_T|.$$

We note that $|X_T| = |X_{S^{-1} \circ T \circ S}|$ for any $S, T \in \mathbb{G}_k$, i.e., $|X_T|$ is constant within each conjugacy class of $\mathbb{G}_k$. Based on the isomorphism between $\mathbb{G}_k$ and the permutation group S_5, we have the following conjugacy class representatives (see Table 2). We denote the identity map as T_0.

Class	Class Rep.	Correspondent in S_5	Class size
0	T_0	Identity	1
1	T_2	(23)	10
2	T_1	(14)(23)	15
3	$T_2 \circ T_4$	(132)	20
4	$T_1 \circ T_4$	(1324)	30
5	G_2	(12345)	24
6	$T_6 \circ T_1 \circ G_2$	(13)(245)	20

TABLE 2. Conjugacy Classes of $\mathbb{G}_k$

We now evaluate $|X_T|$ for each class. Of course, $|X_{T_0}| = |X_k| = (k-1)(k^4 - 9k^3 + 36k^2 - 69k + 51)$ from (2.3). Classes 1 to 4 have been dealt with in [6]. Specifically,

$$|X_{T_2}| = \begin{cases} (k-1)(k-3)^2 & \text{if } k \text{ odd,} \\ (k-1)(k-3)^2 + 6(k-2) & \text{if } k \text{ even,} \end{cases}$$

$$|X_{T_1}| = \begin{cases} k^3 - 5k^2 + 9k - 5 & \text{if } k \text{ odd,} \\ k^3 - 5k^2 + 10k - 7 & \text{if } k \text{ even,} \end{cases}$$

$$|X_{T_2 \circ T_4}| = \begin{cases} k - 1 & \text{if } 3 \nmid k, \\ 3(k-3) & \text{if } 3 \mid k, \end{cases}$$

and

$$|X_{T_1 \circ T_4}| = \begin{cases} 0 & \text{if } k \text{ odd,} \\ (k-1) & \text{if } k \equiv 2 \pmod{4}, \\ (k-3) & \text{if } k \equiv 0 \pmod{4}. \end{cases}$$

We saw $G_2(t_1,t_2,t_3,t_4,t_5) = (t_5 - t_1, t_3, t_3 - t_4, t_3 - t_1, t_2 + t_3 - t_4)$. Hence, $G_2(\vec{t}\,) = \vec{t} \Longleftrightarrow t_1 = t_2 = t_3, t_4 = 0, t_5 = 2t_1$. Therefore, $X_{G_2} = \{(a,a,a,0,2a) \mid a \in \mathbb{Z}_k; a \neq 0\}$ and so $|X_{G_2}| = k-1$.

Now, $T_6 \circ T_1 \circ G_2(t_1,t_2,t_3,t_4,t_5) = (-t_3, t_5 - t_3, t_1 - t_4, t_1 - t_3, -t_2 - t_3 + t_5)$. Hence $T_6 \circ T_1 \circ G_2(\vec{t}\,) = \vec{t} \Longleftrightarrow t_1 = t_2 = -t_3, t_4 = 2t_1, t_5 = 0$. Therefore, $X_{T_6 \circ T_1 \circ G_2} = \{(a,a,-a,2a,0) \mid a \in \mathbb{Z}_k; 3a \neq 0\}$ and so

$$|X_{T_6 \circ T_1 \circ G_2}| = \begin{cases} k-1 & \text{if } 3 \nmid k, \\ k-3 & \text{if } 3 \mid k. \end{cases}$$

So, the number of orbits, by Burnside's theorem, is

$$N_{\mathbb{G}_k} = \frac{1}{120}\Big[(k-1)(k^4 - 9k^3 + 61k^2 - 189k + 280) \\ + \begin{cases} 0 & \text{if } k \equiv 1,5,7,11 \pmod{12}, \\ 40k - 200 & \text{if } k \equiv 3,9 \pmod{12}, \\ 105k - 180 & \text{if } k \equiv 2,10 \pmod{12}, \\ 105k - 240 & \text{if } k \equiv 4,8 \pmod{12}, \\ 145k - 380 & \text{if } k \equiv 6 \pmod{12}, \\ 145k - 440 & \text{if } k \equiv 0 \pmod{12}. \end{cases}\Bigg] \tag{2.4}$$

Table 3 compares $N_{\mathbb{T}_k}$ and $N_{\mathbb{G}_k}$, as calculated by (2.2) and (2.4) respectively, for small k and outlines the percentage reduction in the number of orbits as a result of the action of the bigger group $\mathbb{G}_k$ instead of $\mathbb{T}_k$.

k	$\lvert X_k \rvert$	$N_{\mathbb{T}_k}$	$N_{\mathbb{G}_k}$	Reduction
2	1	1	1	0%
3	12	1	1	0%
4	93	11	6	45%
5	424	28	12	57%
6	1425	92	33	64%
7	3876	207	63	70%
8	9037	466	131	72%
∞				80%

TABLE 3. Reduction in Number of Orbits

We now want to incorporate T_9 and T_{10}. However, T_9 and T_{10} are actually redundant and do not reduce the number of orbits any further. We first note that $T_1(t_1,t_2,t_3,0,t_5) = (t_2,t_1,t_3,0,t_5) = T_9(t_1,t_2,t_3,0,t_5)$. So T_9 is a special case of T_1 and is already included in the action of $\mathbb{T}_k$. Similarly, if we consider the map $f = T_2 \circ T_1 \circ T_6 \circ T_1 \circ T_6 \circ T_1 \circ T_6 \circ T_2 \circ T_8 \circ T_1 \circ T_6 \in \mathbb{G}_k$, then $f(t_1,t_2,t_3,t_4,t_5) = (t_3 - t_5, t_2 - t_5, t_1 - t_5, t_4 - t_5, -t_5)$. Hence, $f(t_1,t_2,t_3,t_4,0) = (t_3,t_2,t_1,t_4,0) = T_{10}(t_1,t_2,t_3,t_4,0)$. So T_{10} is a special case of f and is already included in the action of $\mathbb{G}_k$.

3. Concluding Remarks

Values of finite field hypergeometric functions have also been related to Fourier coefficients of modular forms (see [7] for a survey of these connections). A curious byproduct of the work in [6] is the discovery of new such (conjectural) relations. All these new relations were found by searching the orbits for ${}_3F_2$'s whose bottom line arguments were trivial characters and whose complex conjugate was in the same orbit. See [7] for full details.

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Appendix A. Python Code

The following Python code generates the group $\mathbb{G}_k$ and returns, in separate text files, the orbits for each k in the input range. It is also available on the first author's webpage.

```
import sympy as sym
from sympy import poly

a = sym.Symbol('a')
b = sym.Symbol('b')
c = sym.Symbol('c')
d = sym.Symbol('d')
e = sym.Symbol('e')

def make_T(Fpair): #Subfunction for the group builder
    def T(L):
        return Fpair[0][0](Fpair[1][0](L))
    return T

#Takes in a specific set of maps and genereates the group G_k using those maps as a base.
#The 120 benchmark is an optimization made with foresight that this group only gets that big,
#removing the constraint has no effect other than making group generation longer
def Build_Group(S, base):
    G = []
    out = set()
    i = 0
    for x in S:
        out.add(tuple(x[1]))
        G.append(x)
    while i < len(G) and len(G) < 120:
        for f in G:
            T = make_T([G[i], f])
            res = T(base)
            if not(tuple(res) in out):
                out.add(tuple(res))
                G.append([T, res])
            T = make_T([f, G[i]])
            res = T(base)
            if not(tuple(res) in out):
                out.add(tuple(res))
                G.append([T, res])
        i += 1
    return G

def genXk(k): #Generates the set G_k will act upon
    X_k = set()
    for m in range(1,k):
        for n in range(1,k):
            for l in range(1,k):
                for p in [t for t in range(0,k) if t not in [m,n,l]]:
                    for q in [t for t in range(0,k) if t not in [m,n,l, (m+n+l - p) %k]]:
                        X_k.add((m,n,l,p,q))
    return X_k

#Takes in G_k and an integer k >= 2 and determines the orbits of the action of G_k on X_k
def genorbits(G, k):
    X_k = genXk(k)
    Y = genXk(k)
    orbits = set()
    while len(Y) > 0:
        for x in X_k:
            if not(x in Y):
                continue
            orbitx = set()
            for f in G:
                y = tuple([l % k for l in f[0](list(x))])
                if(y != x and not(y in orbitx)):
                    orbitx.add(y)
                    Y.discard(y)
            orbitx.add(x)
```

```
            orbits.add(tuple(orbitx))
            Y.discard(x)
    return orbits

#Handler for large scale orbit generation
def orbit_lists(G, n, m):
    for k in range(n, m+1):
        print(f"Generating orbit file for k = {k}...")
        O = genorbits(G, k)
        with open("Orbits_k=" + str(k) + ".txt", mode = "w") as file:
            if len(O) == 1:
                file.write(f"The 1 orbit of the action of G_k on X_k for k = {k} is,\n\n")
            else:
                file.write(f"The {len(O)} orbits of the action of G_k on X_k for k = {k} are,\n\n")

            for orb in O:
                file.write(f"The orbit generated by {orb[len(orb)-1]} is listed below and is of size {len(orb)}\n")
                for x in orb:
                    file.write(str(x) + "\n")
                file.write("\n")
    return 1

#main
def main():
    def T_1(L):
        NL = [L[1] - L[3], L[0] - L[3], L[2] - L[3], -L[3], L[4] - L[3]]
        return NL

    def T_2(L):
        NL = [L[0], L[0] - L[3], L[0] - L[4], L[0] - L[1], L[0] - L[2]]
        return NL

    def T_6(L):
        NL = [L[3] - L[0], L[1], L[2], L[3], L[1] + L[2] - L[4]]
        return NL

    def T_8(L):
        NL = [L[0], L[2], L[1], L[4], L[3]]
        return NL

    def T_16(L):
        return T_1(T_6(L))

    base = [a, b, c, d, e]

    S = [[T_2, T_2(base)], [T_16, T_16(base)], [T_8, T_8(base)]]

    G = Build_Group(S, base)
    print("This program will generate orbit files for the action of G_k on X_k for all integers k >=2 in a user given range")
    print("These files will be saved as 'orbit_k=i.txt' as they are generated")
    print("Warning: This code may take exceedingly long to run on some machines for large values of k")
    n = int(input("Enter a starting integer n (this number must be >= 2): "))
    m = int(input("Enter a finishing integer m (only one file will be generated if m = n): "))
    orbit_lists(G, n, m)

main()
```

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