A MATHON-TYPE CONSTRUCTION FOR DIGRAPHS AND IMPROVED LOWER BOUNDS FOR RAMSEY NUMBERS

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Abstract. We construct an edge-colored digraph analogous to Mathon's construction for undirected graphs. We show that this graph is connected to the k-th power Paley digraphs and we use this connection to produce improved lower bounds for multicolor directed Ramsey numbers.

1. Introduction

In [5], Mathon leveraged properties of generalized Paley graphs to improve lower bounds on diagonal multicolor (undirected) Ramsey numbers. He did this by constructing a multicolored graph which contained monochromatic induced subgraphs isomorphic to the generalized Paley graph. Among his results were $R(7,7) \geq 205$, $R(9,9) \geq 565$, $R(10,10) \geq 798$ and $R_3(4) \geq 128$, which are still the best known lower bounds today [9]. Independently, Shearer [13] produced the same results in the two-color case using an equivalent construction. More recently, Xu and Radziszowski [14] made incremental improvements to Mathon's construction and showed that $R_3(7) \geq 3214$ (increased from Mathon's 3211), which is the current best known lower bound.

In this paper, we adapt Mathon's construction to digraphs and leverage properties of k-th power Paley digraphs to produce improved lower bounds for diagonal multicolor directed Ramsey numbers. For the remainder of this paper all Ramsey numbers will be directed, and will be denoted $R_t(m)$. As such, $R_t(m)$ is the least positive integer n such that any tournament with n vertices, whose edges have been colored in t colors, contains a monochromatic transitive subtournament of order m. When $t = 1$ we recover the usual directed Ramsey number $R(m)$, so we drop the subscript in this case. Recall, a tournament is transitive if, whenever $a \to b$ and $b \to c$, then $a \to c$. Our main results, which improve on the previously best known lower bounds, can be summarized as follows.

Theorem 1.1. *$R(8) \geq 57, R(11) \geq 169, R(12) \geq 217, R(14) \geq 401, R(15) \geq 545, R(16) \geq 737, R(17) \geq 889, R(18) \geq 1241, R(19) \geq 1321$ and $R(20) \geq 1945$.*

Theorem 1.2. *For $t \geq 4$,*

$$R_t(3) \geq 169 \cdot 3^{t-4} + 1.$$

For $t \geq 2$,

$$R_t(6) \geq 829 \cdot 27^{t-2} + 1 \qquad \text{and} \qquad R_t(8) \geq 3320 \cdot 56^{t-2} + 1.$$

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2. Preliminaries and Notation

For a graph G, we denote its vertex set by $V(G)$, so the order of G is $\#V(G)$. For a vertex v of a digraph G, we will denote the set of vertices which are out-neighbors of v by $\mathrm{ON}(v)$ and the set of in-neighbors by $\mathrm{IN}(v)$. If the edges of G are colored, we will denote the set of out-neighbors (resp. in-neighbors) of v connected via an edge of color i by $\mathrm{ON}_i(v)$ (resp. $\mathrm{IN}_i(v)$). We define the set of neighbors of v as $\mathrm{N}(v) := \mathrm{ON}(v) \cap \mathrm{IN}(v)$ and the set of color i neighbors as $\mathrm{N}_i(v) := \mathrm{ON}_i(v) \cap \mathrm{IN}_i(v)$. We will refer to any collection of vertices in G, which are pairwise connected via two edges oriented in opposite directions, as a clique, i.e., $C \subseteq V(G)$ is a clique if, for all $u, v \in C$, $u \to v$ and $v \to u$ are edges of G. Further, if all those edges are of color i, we will refer to it as a color i clique.

We note that a tournament of order m is transitive if and only if the set of out-degrees of its vertices is $\{0, 1, \ldots, m-1\}$ [7, Ch. 7]. Thus, we can represent a transitive subtournament of order m by the m-tuple of its vertices $(a_1, a_2, \ldots, a_m)$, listed in order such that the out-degree of vertex a_i is $m - i$, i.e. the corresponding m-tuple of out-degrees is $(m-1, m-2, \ldots, 1, 0)$. We let $\mathcal{K}_m(G)$ denote the number of transitive subtournaments of order m contained in a digraph G.

3. Mathon-Type Construction for Digraphs

Let $k \geq 2$ be an even integer. Let q be a prime power such that $q \equiv k+1 \pmod{2k}$. This condition ensures that -1 is not a k-th power in $\mathbb{F}_q$, the finite field with q elements, but is a $\frac{k}{2}$-th power. Let S_k be the subgroup of the multiplicative group $\mathbb{F}_q^*$ of order $\frac{q-1}{k}$ containing the k-th power residues, i.e., if ω is a primitive element of $\mathbb{F}_q$, then $S_k = \langle \omega^k \rangle$. We define $S_{k,0} := \{0\}$ and $S_{k,i} := \omega^{i-1} S_k$, for $1 \leq i \leq \frac{k}{2}$, so that $S_{k,1} = S_k$. We note that $-S_{k,i} = \omega^{\frac{k}{2}} S_{k,i}$ (as $-1 = \omega^{\frac{q-1}{2}}$ and $\frac{q-1}{2} \equiv \frac{k}{2} \pmod{k}$), yielding the disjoint union

$$\mathbb{F}_q = S_{k,0} \cup \bigcup_{i=1}^{k/2} S_{k,i} \cup \bigcup_{i=1}^{k/2} -S_{k,i}.$$

Let $X := (\mathbb{F}_q \times \mathbb{F}_q) \setminus \{(0,0)\}$. We define an equivalence relation $\sim$ on X where $(a,b) \sim (c,d)$ if $(c,d) = (ag, bg)$ for some $g \in S_k$. We denote the equivalence class of (a,b) by $[a,b]$. There are $n := k(q+1)$ such equivalence classes, each containing $|S_k| = \frac{q-1}{k}$ elements. Let $M_k(q)$ be the edge-colored digraph of order n, with vertex set $X/\sim$, where $[a,b] \to [c,d]$ is an edge in color i, $0 \leq i \leq \frac{k}{2}$, if and only if $bc - ad \in S_{k,i}$. We note that this is well-defined as $gS_{k,i} = S_{k,i}$ for all $g \in S_k$. We also note that any pair of vertices of $M_k(q)$ will either be connected by a single oriented edge in color i, for some $1 \leq i \leq \frac{k}{2}$, or, connected by two edges of color 0 oriented in opposite directions. For ease of illustration in what follows, we will represent the former case by $v_1 \xrightarrow{i} v_2$ and the latter case by $v_1 \xleftrightarrow{0} v_2$.

Proposition 3.1. *$M_k(q)$ is vertex transitive.*

Proof. For $s \in \mathbb{F}_q$, define the maps ρ_s and σ_s on $X/\sim$ by

$$\rho_s : [a,b] \to [a, b + as]$$
$$\sigma_s : [a,b] \to [a + bs, b].$$

It is easy to show that both ρ_s and σ_s are well-defined automorphisms of $M_k(q)$. Let $[a,b]$ and $[c,d]$ be distinct vertices of $M_k(q)$. Assume first that $b,c \neq 0$ and let $s_1, s_2 \in \mathbb{F}_q$ satisfy $a + bs_1 = c$ and $b + cs_2 = d$. Then $\rho_{s_2}(\sigma_{s_1}[a,b]) = [c,d]$. If $b = 0$ then $a \neq 0$, and we can first apply $\rho_1[a,0] = [a,a]$ and then proceed as before. If $c = 0$ then $d \neq 0$, and we can proceed as before to get to $[d,d]$. Then we apply $\sigma_{-1}[d,d] = [0,d]$. □

Proposition 3.2. *For $0 \leq i \leq \frac{k}{2}$, let Γ_i be the subgraph of $M_k(q)$, with vertex set $X/\sim$, induced by the color i edges of $M_k(q)$.*

(1) Γ_0 is the disjoint union of $q+1$ color 0 cliques of order k.
(2) $\Gamma_1, \Gamma_2, \ldots, \Gamma_{\frac{k}{2}}$ are pairwise isomorphic.

Proof. (1) The neighbors of [0,1] in Γ_0 are $\mathrm{N}_0([0,1]) = \{[0,\omega^j] \mid j = 1,2,\ldots,k-1\}$. All elements of $\mathrm{N}_0([0,1])$ are neighbors of each other in Γ_0 and, thus, $[0,1]$ and its neighbors form a clique of order k. As $M_k(q)$ is vertex transitive, every vertex belongs to such a clique. And, as the elements of $\mathrm{N}_0([0,1])$ are not neighbors of any other vertices in Γ_0, all such cliques are disjoint. Therefore, there must be $\frac{n}{k} = q+1$ of them. (2) Γ_i is isomorphic to Γ_{i+1}, for all $1 \leq i \leq \frac{k}{2} - 1$, via the map $[a,b] \to [wa, b]$. □

Proposition 3.3. *Let $v \in V(M_k(q))$. Let $x \in \mathrm{N}_0(v)$. Then for any $i \in \{1,2,\ldots,\frac{k}{2}\}$,*
$$\mathrm{ON}_i(x) \cap \mathrm{ON}_i(v) = \mathrm{IN}_i(x) \cap \mathrm{IN}_i(v) = \emptyset.$$

Proof. As $M_k(q)$ is vertex transitive, it suffices to prove for $v = [0,1]$. Then, let $x \in \mathrm{N}_0([0,1])$, i.e., $x = [0, w^j]$ for some $j = 1,2,\ldots,k-1$. Now
$$[0,\omega^j] \xrightarrow{i} [c,d] \iff \omega^j c \in S_{k,i} \iff c \in \{\omega^{kl+i-j-1} \mid l = 0,1,\ldots,\tfrac{q-1}{k} - 1\},$$
and so
$$\mathrm{ON}_i(x) = \mathrm{ON}_i([0,\omega^j]) = \{[\omega^{i-j-1 \pmod k}, d] \mid d \in \mathbb{F}_q\}.$$
Also,
$$\mathrm{ON}_i(v) = \mathrm{ON}_i([0,1]) = \{[\omega^{i-1}, d] \mid d \in \mathbb{F}_q\}.$$
As $j \not\equiv 0 \pmod k$, we get that $\mathrm{ON}_i(x) \cap \mathrm{ON}_i(v) = \emptyset$. Similar arguments produce
$$\mathrm{IN}_i(x) = \mathrm{IN}_i([0,\omega^j]) = \{[\omega^{i-j-1+\frac{k}{2} \pmod k}, b] \mid b \in \mathbb{F}_q\}$$
and
$$\mathrm{IN}_i(v) = \mathrm{IN}_i([0,1]) = \{[\omega^{i-1+\frac{k}{2}}, b] \mid b \in \mathbb{F}_q\}.$$
So, $\mathrm{IN}_i(x) \cap \mathrm{IN}_i(v) = \emptyset$. □

4. Relation to the k-th power Paley digraphs

Recall from Section 3, $k \geq 2$ is an even integer and q is a prime power such that $q \equiv k+1 \pmod{2k}$. S_k is the subgroup of $\mathbb{F}_q^*$ containing the k-th power residues, i.e., if ω is a primitive element of $\mathbb{F}_q$, then $S_k = \langle \omega^k \rangle$, and $S_{k,i} := \omega^{i-1} S_k$, for $1 \leq i \leq \frac{k}{2}$.

We now recall some definitions and properties from [6] concerning Paley digraphs. We define the *k-th power Paley digraph* of order q, $G_k(q)$, as the graph with vertex set $\mathbb{F}_q$ where $a \to b$ is an edge if and only if $b - a \in S_k$. We note that $-1 \notin S_k$ so $G_k(q)$ is a well-defined oriented graph. For each $1 \leq i \leq \frac{k}{2}$, we define the related directed graph $G_{k,i}(q)$ with vertex set $\mathbb{F}_q$ where $a \to b$ is an edge if and only if $b - a \in S_{k,i}$. Each $G_{k,i}(q)$ is isomorphic to $G_{k,1}(q) = G_k(q)$, the k-th power Paley digraph, via the map $f_i : V(G_k(q)) \to V(G_{k,i}(q))$ given by $f_i(a) = \omega^{i-1}a$. Now consider the *multicolor k-th power Paley tournament* $P_k(q)$ whose vertex set is $\mathbb{F}_q$ and whose edges are colored in $\frac{k}{2}$ colors according to $a \to b$ has color i if $b - a \in S_{k,i}$. Note that the induced subgraph of color i of $P_k(q)$ is $G_{k,i}(q)$. Thus, $P_k(q)$ contains a monochromatic transitive subtournament of order m if and only if $G_k(q)$ contains a transitive subtournament of order m.

Proposition 4.1. *Let $i \in \{1, 2, \ldots, \frac{k}{2}\}$. Let $v \in V(M_k(q))$. Then the induced subgraph of $M_k(q)$ with vertex set $\mathrm{ON}_i(v)$ is isomorphic to $P_k(q)$.*

Proof. As $M_k(q)$ is vertex transitive, it suffices to prove for $v = [0, 1]$. Let H denote the induced subgraph of $M_k(q)$ with vertex set $\mathrm{ON}_i([0, 1])$. In the proof of Proposition 3.3 we saw that $\mathrm{ON}_i([0, 1]) = \{[\omega^{i-1}, d] \mid d \in \mathbb{F}_q\}$. So $\#V(H) = |\mathrm{ON}_i([0, 1])| = q = \#V(P_k(q))$. Now consider the bijective map $\phi : V(H) \to V(P_k(q))$ given by $\phi([\omega^{i-1}, d]) = -\omega^{i-1}d$. It remans to show that ϕ is color-preserving. Let $[\omega^{i-1}, d_1] \in V(H)$ and let $[\omega^{i-1}, d_2] \in \mathrm{ON}_s([\omega^{i-1}, d_1])]$ for some $s \in \{1, 2, \ldots, \frac{k}{2}\}$ (note that $s \neq 0$ otherwise $d_1 = d_2$). Now,

$$\begin{aligned}
[\omega^{i-1}, d_1] \xrightarrow{s} [\omega^{i-1}, d_2] &\iff d_1\omega^{i-1} - \omega^{i-1}d_2 \in S_{k,s} \\
&\iff \phi([\omega^{i-1}, d_2]) - \phi([\omega^{i-1}, d_1]) \in S_{k,s} \\
&\iff \phi([\omega^{i-1}, d_1]) \xrightarrow{s} \phi([\omega^{i-1}, d_2]),
\end{aligned}$$

as required. □

Recall that any pair of vertices of $M_k(q)$ will either be connected by a single oriented edge in color i, for some $1 \leq i \leq \frac{k}{2}$, or, connected by two edges of color 0 oriented in opposite directions. We now replace all these pairs of color 0 edges with a single oriented edge of color $1 \leq i \leq \frac{k}{2}$, where the new color and orientation are randomly assigned. We call this altered graph $M_k^*(q)$, which is a tournament whose edges are colored in $\frac{k}{2}$ colors.

Theorem 4.2. *Let $k \geq 2$ be an even integer and q be a prime power such that $q \equiv k + 1 \pmod{2k}$. Let $m \geq k - 1$ be a positive integer. If $P_k(q)$ contains no monochromatic transitive subtournament of order m, then $M_k^*(q)$ contains no monochromatic transitive subtournament of order $m + 2$.*

Proof. Assume $P_k(q)$ contains no monochromatic transitive subtournament of order m. We note that $0 \xrightarrow{i} \omega^{i-1}$ is an edge in $P_k(q)$ for all $1 \leq i \leq \frac{k}{2}$, and, so, $m \geq 3$ necessarily. Let T_l^* be a monochromatic, in color i, $1 \leq i \leq \frac{k}{2}$, transitive subtournament of $M_k^*(q)$ of order l. We will show that $l < m+2$. We can assume $l \geq 4$, as, otherwise, $l < 4 \leq m+1$, as required. We represent T_l^* by the l-tuple of its vertices $(a_1, a_2, \ldots, a_l)$ with the corresponding l-tuple of out-degrees $(l-1, l-2, \ldots, 1, 0)$. Let T_l be the corresponding subgraph of $M_k(q)$ before the color 0 edges were reassigned, i.e., T_l also has vertices $a_1, a_2, \ldots, a_l$ but some vertices may be connected by two edges of color 0 oriented in opposite directions.

Assume $a_1 \overset{0}{\longleftrightarrow} a_2$ in $M_k(q)$. Consider a_t for $3 \leq t \leq l$. Then there are four possibilities for the triangle (a_1, a_2, a_t) in $M_k(q)$:

$$\begin{tikzcd} a_1 \arrow[rr, leftrightarrow, "0"] \arrow[dr, "i"'] & & a_2 \arrow[dl, "i"] \\ & a_t & \end{tikzcd}$$

(1)

$$\begin{tikzcd} a_1 \arrow[rr, leftrightarrow, "0"] \arrow[dr, "i"'] & & a_2 \\ & a_t \arrow[ur, "0"'] & \end{tikzcd}$$

(2)

$$\begin{tikzcd} a_1 \arrow[rr, leftrightarrow, "0"] & & a_2 \arrow[dl, "i"] \\ & a_t \arrow[ul, "0"] & \end{tikzcd}$$

(3)

$$\begin{tikzcd} a_1 \arrow[rr, leftrightarrow, "0"] & & a_2 \\ & a_t \arrow[ul, "0"] \arrow[ur, "0"'] & \end{tikzcd}$$

(4)

By Proposition 3.3, $\mathrm{ON}_i(a_1) \cap \mathrm{ON}_i(a_2) = \emptyset$ so case (1) can't happen. Now consider case (2). As $M_k(q)$ is vertex transitive, we can let $a_2 = [0, 1]$, without loss of generality. Then $a_1, a_t \in \mathrm{N}_0([0, 1]) = \{[0, \omega^j] \mid j = 1, 2, \ldots, k-1\}$. If we let $a_1 = [0, \omega^{j_1}]$ and $a_t = [0, \omega^{j_2}]$, for some $1 \le j_1 \ne j_2 \le k-1$, then $a_1 \xrightarrow{i} a_t$ implies $0 = \omega^{j_1} \cdot 0 - 0 \cdot \omega^{j_2} \in S_{k,i}$, which is a contradiction. Case (3) is isomorphic to case (2). So, if $a_1 \xleftrightarrow{0} a_2$, then case (4) is the only possibility, which inductively implies that T_l is monochromatic in color 0. Thus, by Proposition 3.2 (1), T_l must be contained in a color 0 clique of Γ_0 and so $l \le k \le m+1$.

Now assume $a_1 \xrightarrow{i} a_2$ in $M_k(q)$. Consider a_t for $3 \le t \le l$. Again, we see that there are four possibilities for the triangle (a_1, a_2, a_t) in $M_k(q)$:

$$\begin{tikzcd} a_1 \arrow[rr, "i"] \arrow[dr, "i"'] & & a_2 \arrow[dl, "i"] \\ & a_t & \end{tikzcd}$$

(i)

$$\begin{tikzcd} a_1 \arrow[rr, "i"] \arrow[dr, "i"'] & & a_2 \\ & a_t \arrow[ur, "0"'] & \end{tikzcd}$$

(ii)

$$\begin{tikzcd} a_1 \arrow[rr, "i"] & & a_2 \arrow[dl, "i"] \\ & a_t \arrow[ul, "0"] & \end{tikzcd}$$

(iii)

$$\begin{tikzcd} a_1 \arrow[rr, "i"] & & a_2 \\ & a_t \arrow[ul, "0"] \arrow[ur, "0"'] & \end{tikzcd}$$

(iv)

Case (ii) can't happen because $\mathrm{IN}_i(a_2) \cap \mathrm{IN}_i(a_t) = \emptyset$, by Proposition 3.3. Case (iv) is isomorphic to case (2) above, which we've seen is not possible. We now examine case (iii). As $M_k(q)$ is vertex transitive, we can let $a_1 = [0, 1]$, without loss of generality. Then $a_2 \in \mathrm{ON}_i([0, 1]) = \{[\omega^{i-1}, d] \mid d \in \mathbb{F}_q\}$ and $a_t \in \mathrm{N}_0([0, 1]) = \{[0, \omega^j] \mid j = 1, 2, \ldots, k-1\}$. Further,

$$\begin{aligned}
a_2 \xrightarrow{i} a_t &\iff [\omega^{i-1}, d] \xrightarrow{i} [0, \omega^j] \\
&\iff d \cdot 0 - \omega^{i-1} \cdot \omega^j \in S_{k,i} \\
&\iff \omega^{i+j-1} \in -S_{k,i} = \{\omega^{kv+i-1+\frac{k}{2}} \mid v = 0, 1, \ldots, \tfrac{q-1}{k} - 1\} \\
&\iff \omega^j \in \{\omega^{kv+\frac{k}{2}} \mid v = 0, 1, \ldots, \tfrac{q-1}{k} - 1\} \\
&\iff j = \tfrac{k}{2} \\
&\iff a_t = [0, \omega^{\frac{k}{2}}] = [0, -1]
\end{aligned}$$

So, case (iii) is possible but there is only one possible a_t, which means there is only one value of $t \in \{3, \ldots, l\}$ for which $a_1 \xleftrightarrow{0} a_t$. So assume there is an $s \in \{3, \ldots, l\}$ such that

$$\begin{tikzcd} a_1 \arrow[rr, "i"] & & a_2 \arrow[dl, "i"] \\ & a_s \arrow[ul, "0"] & \end{tikzcd}$$

Then $a_1 \xrightarrow{i} a_t$ for all $t \in \{3, \dots, l\} \setminus \{s\}$ and by previous arguments we must have

$$\begin{tikzcd} a_1 \arrow[rr, "i"] \arrow[dr, "i"'] & & a_2 \arrow[dl, "i"] \\ & a_t & \end{tikzcd}$$

Therefore, if $t_1, t_2 \in \{3, \dots, l\} \setminus \{s\}$ with $t_1 < t_2$, then

$$\begin{tikzcd} a_1 \arrow[rr, "i"] \arrow[dr, "i"'] & & a_{t_1} \arrow[dl, leftrightarrow, "0"] \\ & a_{t_2} & \end{tikzcd}$$

is not possible, by Proposition 3.3, and so $a_{t_1} \xrightarrow{i} a_{t_2}$. Thus, if we remove a_s from T_l we get a monochromatic, in color i, transitive subtournament of $M_k(q)$ of order $l-1$, which we call T_{l-1}. Furthermore, $T_{l-1} \setminus \{a_1\}$ is a monochromatic, in color i, transitive subtournament of $M_k(q)$ of order $l-2$. If we let H denote the induced subgraph of $M_k(q)$ with vertex set $\mathrm{ON}_i(a_1)$, then by Proposition 4.1, $T_{l-1} \setminus \{a_1\} \subseteq H \cong P_k(q)$. So, if $P_k(q)$ contains no monochromatic transitive subtournament of order m, then $l-2 < m$.

If there is no $3 \leq t \leq l$ for which (a_1, a_2, a_t) satisfies cases (ii), (ii) or (iv) then all a_t, for $3 \leq t \leq l$, satisfy case (i). Then $a_{t_1} \xrightarrow{i} a_{t_2}$ for all $3 \leq t_1 < t_2 \leq l$ by previous arguments. So, in this case, T_l itself is a monochromatic, in color i, transitive subtournament of $M_k(q)$. Letting H denote the induced subgraph of $M_k(q)$ with vertex set $\mathrm{ON}_i(a_1)$ and, again, using Proposition 4.1, we get that $T_l \setminus \{a_1\} \subseteq H \cong P_k(q)$. So, if $P_k(q)$ contains no monochromatic transitive subtournament of order m, then $l-1 < m$.

Overall, if $P_k(q)$ contains no monochromatic transitive subtournament of order m, then $M_k^*(q)$ contains no monochromatic transitive subtournament of order $m+2$. □

Corollary 4.3. *Let $k \geq 2$ be an even integer and q be a prime power such that $q \equiv k+1 \pmod{2k}$. If $\mathcal{K}_m(G_k(q)) = 0$, for $m \geq k-1$, then $R_{\frac{k}{2}}(m+2) \geq k(q+1)+1$.*

Proof. By definition, $\mathcal{K}_m(G_k(q)) = 0$ means that $G_k(q)$ contains no transitive subtournaments of order m. By the discussion at the start of this section, this implies $P_k(q)$ contains no transitive subtournaments of order m [6]. Consequently, by Theorem 4.2, $M_k^*(q)$ contains no monochromatic transitive subtournament of order $m+2$. Recall, $M_k^*(q)$ is a tournament of order $n = k(q+1)$ whose edges are colored in $\frac{k}{2}$ colors, so $R_{\frac{k}{2}}(m+2) \geq k(q+1)+1$. □

5. Proofs of Theorems 1.1 and 1.2

We now examine properties of $G_k(q)$ and apply Corollary 4.3 to get improved lower bounds for certain directed Ramsey numbers, proving Theorems 1.1 and 1.2.

Proof of Theorem 1.1. Theorem 1.1 corresponds to the case when $k = 2$. For all appropriate $q \leq 1583$ we found, by computer search (see Section 6 for details), the order of the largest transitive subtournament of $G_2(q)$. Then, from this data, we identified the largest q such that $\mathcal{K}_m(G_k(q)) = 0$, for each $3 \leq m \leq 20$. Call this q_m. Then, by definition, $R(m) \geq q_m + 1$. Combining with Corollary 4.3, when $k = 2$, yields $R(m+2) \geq \max(2(q_m + 1) + 1, q_{m+2} + 1)$. The results for $7 \leq m \leq 20$ are shown in Table 1. ($R(m)$ for $3 \leq m \leq 6$ are already known, specifically $R(3) = 4$, $R(4) = 8$ [2], $R(5) = 14$ [10], $R(6) = 28$ [11].) We note that $q_6 = 27$.

m	7	8	9	10	11	12	13	14	15	16	17	18	19	20
q_m	27	47	83	107	107	199	271	367	443	619	659	971	1259	1571
$R(m) \geq$	28	**57**	*84*	*108*	**169**	**217**	*272*	**401**	**545**	**737**	**889**	**1241**	**1321**	**1945**

TABLE 1. Lower Bounds for $R(m)$.

The values of q_m in Table 1, for $7 \leq m \leq 18$, confirm those of Sanchez-Flores [12], and, for $m = 19$, that of Exoo [3]. The best known lower bound for $m = 7$ is $R(7) \geq 34$, due to Neiman, Mackey and Heule [8]. For $8 \leq m \leq 10$ and $12 \leq m \leq 19$ the previously best known lower bound was $R(m) \geq q_m + 1$ [3]. Also from [3] we have that $R(11) \geq 112$. So the values in bold in Table 1 represent an improvement to the previously best known lower bounds, establishing Theorem 1, and the values in italics equal the best known lower bounds. □

Proof of Theorem 1.2. We also performed a similar exercise for $k = 4, 6, 8$ and 10, identifying, in each case, the largest q such that $\mathcal{K}_m(G_k(q)) = 0$, for $3 \leq m \leq 10$. We will denote such q as $q_{m,k}$. Table 2 outlines these values. The values in the last row of the table indicate the upper limit for q in our search. Note that values of $q_{m,k}$ close to this limit will not be optimal.

m	$k=4$	$k=6$	$k=8$	$k=10$
3	13	43	169	71
4	125	343	953	3331
5	157	859	2809	6791
6	829	4339	15625	33191
7	709	4423	26153	43411
8	1709	18523	29929	58771
9	3517	29611	29929	59951
10	7573	29959	29929	59971
$q <$	10000	30000	30000	60000

TABLE 2. Largest q found such that $\mathcal{K}_m(G_k(q)) = 0$.

Now, by definition,

$$R_{\frac{k}{2}}(m) \geq q_{m,k} + 1 \tag{5.1}$$

and, by Corollary 4.3,

$$R_{\frac{k}{2}}(m+2) \geq k(q_{m,k} + 1) + 1 \tag{5.2}$$

when $m \geq k - 1$. We note also that for $t \geq 2$ [4, Prop. 5]

$$R_t(m) \geq (R_{t-1}(m) - 1)(R(m) - 1) + 1. \tag{5.3}$$

It is already known that $R(3) = 4$, $R(4) = 8$ [2], $R(5) = 14$ [10], $R(6) = 28$ [11], $R(7) \geq 34$ [8], $R_2(3) = 14$ [1], $R_2(4) \geq 126$ and $R_3(3) \geq 44$ [6]. We combine all this information, including values from Table 1, to get lower bounds on the Ramsey numbers $R_t(m)$ for $t \geq 2$ and $3 \leq m \leq 10$. The results are shown in Table 3.

For example, in the case $m = 3$, it is already known that $R_2(3) = 14$ [1]. It is also known that $R(3) = 4$ [2], so by (5.3) we get that $R_3(3) \geq (R_2(3) - 1)(R(3) - 1) + 1 = 40$. But, from Table 2, we see that $q_{3,6} = 43$ and so $R_3(3) \geq 44$ by (5.1) which is better. When $t = 4$, (5.3) tells us that $R_4(3) \geq (R_3(3) - 1)(R(3) - 1) + 1 \geq 130$, but (5.1) produces $R_4(3) \geq 170$, as $q_{3,8} = 169$ from Table 2. For $t \geq 5$, (5.3) produces the best bound, i.e., $R_t(3) \geq 169 \cdot 3^{t-4} + 1$. We note that, as $m = 3$, the bound produced by Corollary 4.3, (5.2), is not applicable for $t = \frac{k}{2} > 2$.

In contrast, in the case $m = 8$, (5.2) produces the best bound when $t = 2$. From Table 2, we see that $q_{8,4} = 1709$ and so (5.1) yields $R_2(8) \geq 1710$. From Table 1, we get that $R(8) \geq 57$ and so $R_2(8) \geq (57 - 1)^2 + 1 = 3137$ by (5.3). Again from Table 2, we see that $q_{6,4} = 829$ and so (5.2) yields $R_2(8) \geq 4(829 + 1) + 1 = 3321$, which is better than the bounds coming from both (5.1) and (5.3). For $m = 8$ and $t \geq 3$, (5.3) produces the best bound, i.e., $R_t(8) \geq 3320 \cdot 56^{t-2} + 1$.

The remainder of Table 3 is produced similarly.

<table>
<tr><th>m</th><th>$t=2$</th><th>$t=3$</th><th>$t=4$</th><th>$t \geq 5$</th></tr>
<tr><td>3</td><td>14</td><td>44</td><td>170</td><td>$169 \cdot 3^{t-4} + 1$</td></tr>
<tr><td>4</td><td>126</td><td colspan="3">$125 \cdot 7^{t-2} + 1$</td></tr>
<tr><td>5</td><td colspan="4">$13^t + 1$</td></tr>
<tr><td>6</td><td>830</td><td colspan="3">$829 \cdot 27^{t-2} + 1$</td></tr>
<tr><td>7</td><td colspan="4">$33^t + 1$</td></tr>
<tr><td>8</td><td>3321</td><td colspan="3">$3320 \cdot 56^{t-2} + 1$</td></tr>
<tr><td>9</td><td colspan="4">$83^t + 1$</td></tr>
<tr><td>10</td><td colspan="4">$107^t + 1$</td></tr>
</table>

TABLE 3. Lower bounds for $R_t(m)$.

The general formulas in the cases $m = 3, 6, 8$ improve on what was previously known and establish Theorem 1.2. We note that the $m = 8$ case is the only one where Corollary 4.3 influences the results. For $m \neq 3, 6, 8$, the bounds in Table 3 reflect already known bounds combined with (5.3). $\square$

6. A note on the computer search

In order to use the results of Section 4 to obtain various lower bounds, the central problem is to find a maximum length subtournament of a given directed graph G. For this, we adopt a straightforward recursive approach. Begin with $M \leftarrow 0$ and $T \leftarrow \emptyset$. Given a (possibly empty) transitive subtournament T of G, enumerate $T = \{a_1, \ldots, a_\ell\}$ with $a_i \to a_j$ for all $1 \leq i < j \leq \ell$. Determine the set $S = \bigcap_{i=1}^{\ell} \mathrm{ON}(a_i)$ of possible successors of a_ℓ, where the empty intersection is taken as $V(G)$. If S is empty, set $M \leftarrow \max\{M, \ell\}$; otherwise, for each $s \in S$, recursively apply this procedure to $T \cup \{s\}$. Several obvious optimizations are employed, but this is the essential idea.

We then appeal to Lemma 4.2(c) from [6]. Let $H_k(q)$ be the subgraph of $G_k(q)$ induced by S_k, and let $H_k^1(q)$ be the subgraph of $H_k(q)$ induced by ON(1). By that lemma, $G_k(q)$ has a transitive subtournament of order m if and only if $H_k^1(q)$ has a transitive subtournament of order $m-2$. We therefore apply the recursive procedure described above to the smaller directed graph $H_k^1(q)$, and use that to determine the maximum length transitive subtournament of $G_k(q)$. The full source code used to generate the computational results is available on GitHub[1].

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[1] `https://github.com/AssociateDeadWood/GenPaley`