Solution to Problem 10.5 # 25

25 a. Since the car moves in a circle, its tangential acceleration is 0. The force required to keep the car moving in a circle has magnitude

$$m\kappa\left(\frac{ds}{dt}\right)^2 = \frac{m}{\rho}\left(\frac{ds}{dt}\right)^2$$

where $m = W/g$ is the car mass and $\rho = 1/\kappa$. We want this force to be balanced by the frictional force keeping the car on the road, so

$$\frac{m}{\rho}\left(\frac{ds}{dt}\right)^2 = \mu W = \mu(mg)$$

The m's cancel (so the weight does not matter), and for $\mu = 0.47$, $\rho = 150$, and $g = 32$, the maximum safe speed is

$$\frac{ds}{dt} = \sqrt{\mu g \rho} = \sqrt{(0.47)(32)(150)} = 47 ft/s$$

(about 32 mi/h).

b. With a roadway banked at angle θ, the horizontal force pulling the car toward the center of the circle has magnitude

$$\begin{aligned} ||\mathbf{F}_k|| &= ||\mathbf{F}_N||\sin(\theta) + ||\mathbf{F}_s||\cos(\theta) \\ &= ||\mathbf{F}_N||\sin(\theta) + \mu||\mathbf{F}_N||\cos(\theta) \end{aligned}$$

where $\mathbf{F}_N$ is the normal force and $\mathbf{F}_s$ is the friction force, which is directed downward along the bank. We also have

$$\begin{aligned} mg &= ||\mathbf{F}_N||\cos(\theta) - ||\mathbf{F}_s||\sin(\theta) \\ &= ||\mathbf{F}_N||\cos(\theta) - \mu||\mathbf{F}_N||\sin(\theta) \end{aligned}$$

so that

$$||\mathbf{F}_N|| = \frac{mg}{(\cos(\theta) - \mu\sin(\theta))}$$

and

$$\begin{aligned} ||\mathbf{F}_k|| &= (\sin(\theta) + \mu\cos(\theta))||\mathbf{F}_N|| \\ &= \frac{mg(\sin(\theta) + \mu\cos(\theta))}{\cos(\theta) - \mu\sin(\theta)} \end{aligned}$$

To find the optimal banking speed we set this expression for $||\mathbf{F}_k||$ equal to

$$\begin{aligned}
\frac{m}{\rho}\left(\frac{ds}{dt}\right)^2 &= \frac{mg(\sin(\theta)+\mu\cos(\theta))}{\cos(\theta)-\mu\sin(\theta)} \\
\frac{ds}{dt} &= \sqrt{\frac{\rho g(\sin(\theta)+\mu\cos(\theta))}{\cos(\theta)-\mu\sin(\theta)}}
\end{aligned}$$

For $\mu = 0.47$, $\rho = 150$, $g = 32$ and $\theta = 17^\circ$, we have $\dfrac{ds}{dt} = 65.94$ (about 45 mi/h).

c. For the optimal safe speed to be 50 mi/h (73.33 ft/s) we want θ to satisfy

$$\begin{aligned}
(73.33)^2 &= \frac{150(32)(\sin(\theta)+0.47\cos(\theta))}{\cos(\theta)-0.47\sin(\theta)} \\
\theta &\approx 23.1^\circ
\end{aligned}$$

Solution to Problem 10.5 # 33

The curve $x = \dfrac{y^2}{120}$ has parametric form :

$$\begin{aligned}
x &= \frac{t^2}{120}, \quad y = t \\
x' &= \frac{t}{60}, \quad y' = 1 \\
x'' &= \frac{1}{60}, \quad y'' = 0
\end{aligned}$$

The curvature is

$$\begin{aligned}
\kappa &= \frac{|x'y'' - y'x''|}{[(x')^2 + (y')^2]^{3/2}} \\
&= \frac{|\frac{t}{60}(0) - (1)\frac{1}{60}|}{[(\frac{t}{60})^2 + 1]^{3/2}} \\
&= \frac{60^2}{[t^2 + 60^2]^{3/2}}
\end{aligned}$$

At $t = 0$, $\kappa(0) = \dfrac{1}{60}$ and $A_N = \kappa\left(\dfrac{ds}{dt}\right)^2 = \dfrac{1}{60}\left(\dfrac{ds}{dt}\right)^2$

Thus the maximum safe speed $\dfrac{ds}{dt}$ satisfies

$$
\begin{aligned}
A_N &= \frac{1}{60}\left(\frac{ds}{dt}\right)^2 = 30 \\
\frac{ds}{dt} &= 42.43 \text{ units/s}
\end{aligned}
$$