Transitive subgroups of S_4

The Galois group G of an irreducible polynomial f of degree 4 over F permutes all the 4 different roots of f and therefore it has to be a transitive subgroup of S_4. The only posibilities for such a group are:

1. S_4.

2. A_4 (by one of your exam practice problems, $G = A_4$ if and only if the discriminant $D(f)$ is a square in F).

3. The dihedral group of order 8 (symmetries of the quadrilateral) $D_8 = \{1, (1324), (12)(34), (1423), (13)(24), (14)(23), (12)(34)\}$ and its conjugates.

4. The Klein 4-group, denoted by V_4 (due to the german word *Vier-gruppe* which means "4-group"), is the group of order 4 with multiplication table

$\cdot$	1	a	b	c
1	1	a	b	c
a	a	1	c	b
b	b	c	1	a
c	c	b	a	1

The only nontrivial subgroups of V_4 are $\langle a \rangle$, $\langle b \rangle$ and $\langle c \rangle$. They are all of order 2 (and thus, of index 2 in V_4). Notice that although $|V_4| = 4$ it is not isomorphic to the cyclic group of order 4, C_4 (only one element has order 2 in C_4).

On the other hand, V_4 is a subgroup of the dihedral group D_8.

As a subgroup of S_4, $V_4 = \{1, (12)(34), (13)(24), (14)(23)\}$. Since it contains all the 22-cycles in S_4 it is a normal subgroup.

5. $C_4 = \{1, (1234), (13)(24), (1432)\}$ and its conjugates.