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Differential 'Galois' extensions with new constants

Extensions différentielles « galoisiennes » avec nouvelles constantes

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ABSTRACT

Let F be a differential field with algebraically closed field of constants $\mathcal{C}$ and let E be a differential field extension of F. The field E is a differential Galois extension if it is generated over F by a full set of solutions of a linear homogeneous differential equation with coefficients in F and if its field of constants coincides with $\mathcal{C}$. We study the differential field extensions of F that satisfy the first condition but not the second.

RÉSUMÉ

Soit F un corps différentiel dont le corps des constantes $\mathcal{C}$ est algébriquement clos et soit $E \supset F$ une extension de corps différentiels. Le corps différentiel E est une extension galoisienne différentielle de F s'il est engendré sur F par une base de solutions d'une équation différentielle linéaire homogène à coefficients dans F et si son corps des constantes est $\mathcal{C}$. Nous étudions les extensions différentielles de F qui satisfont la première condition et non la seconde.

1. Introduction

Let F be a differential field of characteristic zero. Let $\mathcal{C}$ denote the field of constants of F (assumed algebraically closed). Let $L = Y^{(n)} + a_1 Y^{(n-1)} + \cdots + a_0 Y$ be a linear differential operator over F.

A Picard–Vessiot, or differential Galois, extension E of F for L is a differential field extension, also with field of constants $\mathcal{C}$ (i.e. a *no new constant extension*), generated over F as a differential field by n solutions of $L = 0$ linearly independent over $\mathcal{C}$ (i.e. *a full set of solutions*). The differential Galois group $G(E/F)$ is the group of differential field automorphisms of E fixing F, and the resulting Galois correspondence includes the fact that the fixed field of E under $G(E/F)$ is F. The same extension E is a Picard–Vessiot extension of F for many different operators L. One way to suppress the explicit reference to a specific operator is to note that a differential field extension $E \supset F$ is a Picard–Vessiot extension for some operator L provided that

(1) There is a group G of differential automorphisms of E over F whose fixed field is F.
(2) E is generated over F as a differential field by a G-submodule V which is finite dimensional over $\mathcal{C}$.
(3) E is a no new constant extension of F.

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The operator L for which E is Picard–Vessiot is produced from V using Wronskian determinants, and then V has a full set of solutions of L as a $\mathcal{C}$-basis (see [2]).

There are natural situations, however, for example the case of certain derivations on coordinate rings of connected algebraic groups, which lead to extensions $E \supset F$ which meet the first two criteria but not the third. Such extensions are our subject here.

We reformulate the situation in terms of rings generated by solutions.

The ring $F[y_{ij}\colon 1 \leqslant i \leqslant n, 0 \leqslant j \leqslant n-1][w^{-1}]$, where $w = \det(y_{ij})$, with derivation $y'_{ij} = y_{i,j+1}$, $y'_{i,n-1} = -a_{n-1}y_{i,n-1} - \cdots - a_0 y_{i0}$ contains and is differentially generated by n solutions y_{i0} to the differential equation $L = 0$ whose Wronskian is a unit, and the same remains true for its homomorphic images. (This is why it is called a *full universal solution algebra* for L in [2].) If we take a maximal differential ideal of the full universal solution algebra and mod it out, the quotient field of the resulting integral domain is a Picard–Vessiot extension of F for L: it has the additional property that its constants are $\mathcal{C}$ as well. It doesn't matter which maximal differential ideal we choose: all are conjugate under the (differential) action of $\mathrm{GL}_n(\mathcal{C})$ extended from the linear action on the y_{ij} [2]. And the stabilizer in $\mathrm{GL}_n(\mathcal{C})$ of the chosen maximal ideal, which acts naturally mod it and on the quotient field, is the group of all differential automorphisms and the differential Galois group over F.

We focus on what happens when a not necessarily maximal prime differential ideal P of the full universal solution algebra is selected, modded out, and we consider the resulting quotient field extension E of F. When the prime ideal is not maximal, the constant field K of E is a proper extension of F, and of course need not be algebraically closed. Our main result shows that nonetheless E is much like a Picard–Vessiot extension of the compositum field FK. In particular, we find an algebraic subgroup G of $\mathrm{GL}_n(K)$ with $E^G = FK$.

Unless (0) is a maximal differential ideal of the full universal solution algebra, the latter will always have non-maximal prime differential ideals to which our results apply. A typical situation to which our results apply is the following: Let H be a connected algebraic group over $\mathcal{C}$, let $D_1, \ldots, D_m$ be a basis of $\mathrm{Lie}(H)$, let $b_i \in F$ and consider the derivation $D = D_F \otimes 1 + \sum b_i \otimes D_i$ on $F[H] = F \otimes \mathcal{C}[H]$. We show that there is an operator L as above which has a full set of solutions in $F(H)$ (the quotient field of $F[H]$) with invertible Wronskian, so that the algebra they differentially generate over F fits into our theory above. The appropriate consequences are drawn.

We will preserve throughout the notational conventions of this introduction. Another useful reference on the Picard–Vessiot theory is [3].

2. Modulo non-maximal primes

Let P be a prime differential ideal of the full universal solution algebra $F[y_{ij}\colon 1 \leqslant i \leqslant n, 0 \leqslant j \leqslant n-1][w^{-1}]$ for $L = Y^{(n)} + a_1 Y^{(n-1)} + \cdots + a_0 Y$, let R_F denote $F[y_{ij}][w^{-1}]$, and let E denote the quotient field of R_F/P. Let K denote the field of constants of E. We choose an algebraic closure $\overline{E}$ of E. Then $\overline{K}$, the algebraic closure of K in $\overline{E}$, is also algebraically closed and is the field of constants of the composita $E\overline{K}$ and $F\overline{K}$. To see that $\overline{K}$ is indeed the field of constants, we use the fact that $E\overline{K} \supseteq E$ is an algebraic extension, so that any constant must be algebraic over the constants K of E.

Lemma 1. *$E\overline{K}$ is a Picard–Vessiot extension of $F\overline{K}$ for L.*

Proof. $E\overline{K}$ is generated over $F\overline{K}$ by a full set of solutions, with non-zero Wronskian of $L = 0$, so the only issue is new constants. As we just noted, the constants of $E\overline{K}$ are $\overline{K}$ which is also the constants of $F\overline{K}$. □

Let E_0 be a differential subfield of E containing F and K, for example FK. Then Lemma 1 also implies that $E\overline{K}$ is a Picard–Vessiot extension of $E_0\overline{K}$.

Let z_{ij} be the image of y_{ij} in $E\overline{K}$. Let $R_{E_0\overline{K}} = E_0\overline{K} \otimes_F R_F$ and consider the homomorphism $R_{E_0\overline{K}} \to E\overline{K}$ induced from $y_{ij} \mapsto z_{ij}$. Let M be its kernel. Note that $R_{E_0\overline{K}}$ is also a full universal solution algebra for L over $E_0\overline{K}$. If M were not maximal then as above the quotient field $E\overline{K}$ of $R_{E_0\overline{K}}/M$ would contain a new constant, a contradiction. So M is a maximal differential ideal.

Let $R_{E_0} = E_0 \otimes_F R_F$ and let M_0 be the kernel of the homomorphism $R_{E_0} \to E$ induced from $y_{ij} \mapsto z_{ij}$. We have $R_{E_0\overline{K}} \supseteq R_{E_0}$ and $M \cap R_{E_0} = M_0$.

We note that $\overline{K} \cap E = K$, since anything algebraic over constants is a constant. It follows that $\overline{K} \otimes_K E$ is an integral domain, and hence so is $\overline{K} \otimes_K E_0$. Since these are integral domains algebraic over fields, they are themselves fields. Thus $E\overline{K} = \overline{K} \otimes_K E$ and $E_0\overline{K} = \overline{K} \otimes_K E_0$.

Lemma 2. *$E_0\overline{K} \cap E = E_0$.*

Proof. From the above equalities, the assertion is the obvious one that $(\overline{K} \otimes_K E_0) \cap (K \otimes_K E) = K \otimes_K E_0$ in $\overline{K} \otimes_K E$. □

These observations allow us to conclude that the ideal M is induced (we use the above notation):

Lemma 3. *$\overline{K} \otimes_K R_{E_0}/M_0$ is isomorphic to $R_{E_0\overline{K}}/M$ as $\overline{K}$-algebra. In particular*:

(1) *M_0 is a maximal differential ideal.*
(2) *$M = R_{E_0\overline{K}} M_0$ as ideals.*
(3) *$M = \overline{K} M_0$ as $\overline{K}$-vector spaces.*

Proof. By the discussion preceding the lemma, the subring $S = \overline{K} \otimes_K R_{E_0}/M_0$ of $\overline{K} \otimes_K E$ can be regarded as a subring of $E\overline{K}$. Viewed in that way, it is the $E_0\overline{K}$ algebra generated by z_{ij} and the inverse Wronskian, and that algebra is $R_{E_0\overline{K}}/M$. This implies the main assertion of the lemma and the others are direct consequences. □

The rings R_{E_0} and $R_{E_0\overline{K}}$ are generated by the vector spaces $V_K = \sum_{i,j} K y_{ij}$ and $V_{\overline{K}} = \sum_{i,j} \overline{K} y_{ij}$ (and the inverse Wronskians) over their coefficient fields. Differential actions of the groups $\mathrm{GL}_n(K)$ and $\mathrm{GL}_n(\overline{K})$ on R_{FK} and $R_{F\overline{K}}$ come from their actions on these vector spaces [2]. The inclusion $\mathrm{GL}_n(K) \subseteq \mathrm{GL}_n(\overline{K})$ is compatible with the inclusion $V_K \subseteq V_{\overline{K}}$. So we can regard $\mathrm{GL}_n(K)$ as acting on $R_{E_0\overline{K}}$, and the restriction of that action to the subring R_{E_0} is the given action. The stabilizer $\mathrm{GL}_n(\overline{K})_M$ of the ideal M is an algebraic subgroup of $\mathrm{GL}_n(\overline{K})$; in fact, it is the differential Galois group $G(E\overline{K}/E_0\overline{K})$ [2].

Lemma 4. *The stabilizer $\mathrm{GL}_n(K)_{M_0}$ of M_0 is Zariski dense in $\mathrm{GL}_n(\overline{K})_M$.*

Proof. We have by Lemma 3 that $M = \overline{K} M_0$. If M_0 were a finite dimensional K-vector space, the result would be obvious. Since the actions here are rational, we can reduce immediately to the finite dimensional case and conclude the same result. □

We are now ready for the main result.

Theorem 1. *Let $G(E/E_0) = \mathrm{GL}_n(K)_{M_0}$. Then $E^{G(E/E_0)} = E_0$.*

Proof. Because $G(E/E_0)$ is Zariski dense in $G(E\overline{K}/E_0\overline{K})$ by Lemma 4, and $E\overline{K} \supset E_0\overline{K}$ is Picard–Vessiot by Lemma 1 we have $(E\overline{K})^{G(E/E_0)} = E_0\overline{K}$. It follows that $E^{G(E/E_0)} \subseteq E_0\overline{K}$. So it suffices to show that $E \cap E_0\overline{K} = E_0$. This is just Lemma 2. □

3. Derivations of group coordinate rings

Warning: *the symbol E is used in this section differently than above.*

Let H be a connected linear algebraic group over $\mathcal{C}$, let $D_1, \ldots, D_m$ be a basis of Lie(H), let $b_i \in F$ and consider the derivation $D = D_F \otimes 1 + \sum 1 \otimes D_i$ on $F[H] = F \otimes \mathcal{C}[H]$. We regard H as acting on $F[H]$ by left translations $(h \cdot f(g) = f(gh))$ so that D is H-equivariant. Both D and the H-action extend to the quotient field $F(H)$ and commute with each other.

Let W be some finite dimensional $\mathcal{C}$-subspace which is H-stable and generates $\mathcal{C}[H]$ as a $\mathcal{C}$-algebra. Then the following properties hold for the differential field extension $E = F(H) \supset F$:

(1) The group H is a group of differential automorphisms of E over F such that $E^H = F$.
(2) There is a finite dimensional H-stable, $\mathcal{C}$-vector space $W \subset E$ such that $E = F\langle W\rangle$ is differentially generated over F by W.

As remarked in the introduction, if additionally the constants of E were those of F, E would be a Picard–Vessiot extension of F. We do not make that assumption here; hence any field extension meeting the above two criteria is called a *pre-Picard–Vessiot* (briefly *pPV*) extension of F. For example, if X is an irreducible H-torsor and D a derivation on $F(X)$ induced by an element of the corresponding twisted Lie algebra (see [1] for details) then $F(X)$ is a *pPV* extension of F.

Let K be the field of constants of an arbitrary *pPV* extension E. Let $V = KW$, where W is as defined above except that we can replace $\mathcal{C}[H]$ by the coordinate ring $\mathcal{C}[X]$ of an irreducible H-torsor, and let $z_1, \ldots, z_m$ be a K-basis of W. Note that the Wronskian $w(z_1, \ldots, z_m)$ is non-zero by construction. H acts on KW, although not K-linearly in general. Nonetheless, for $h \in H$ we have a matrix $\alpha(h) \in \mathrm{GL}_m(K)$ such that

$$\left(z_1^h, \ldots, z_m^h\right) = (z_1, \ldots, z_m)\alpha(h).$$

We can differentiate both sides of this equation and obtain

$$\left(\left(z_1'\right)^h, \ldots, \left(z_m'\right)^h\right) = \left(z_1', \ldots, z_m'\right)\alpha(h)$$

using the fact that K is the field of constants. Repeated differentiation shows that a similar formula holds for higher derivatives as well.

Now suppose Y is a differential indeterminate over F, and consider the Wronskian determinant $w(Y, z_1, \ldots, z_n)$ in $E\{Y\}$. When we expand this determinant along the first column, the various minors that occur have rows of the form $(z_1^{(i)}, \ldots, z_m^{(i)})$. These rows transform under $h \in H$ as above via multiplication by $\alpha(h)$, and hence the minors transform via multiplication by $\det(\alpha(h))$. This applies to the coefficient $w(z_1, \ldots, z_m)$ of $Y^{(m)}$ as well. It then follows that the coefficients of $L = w(z_1, \ldots, z_m)^{-1} w(Y, z_1, \ldots, z_n) = Y^{(m)} + a_1 Y^{(m-1)} + \cdots + a_m Y^{(0)}$ are invariant under any $h \in H$ and hence lie in F. (This is an adaptation of the argument in [2].)

We thus have a homomorphism from the full universal solution algebra $R_F = F[y_{ij}][w^{-1}]$ over F for L to E by $y_{i0} \mapsto z_i$ whose kernel is a prime ideal P. Let E_1 denote the quotient field of its image. By construction, F is a subfield of E_1, and $E = E_1 K$. The constants of E_1 are $K_1 = K \cap E_1$. By Theorem 1, we have $FK_1 = E_1^{G(E_1/FK_1)}$, where $G(E_1/FK_1)$ is a subgroup of $\mathrm{GL}_m(K_1)$. Thus the extension $E \supset F$ breaks into the subextensions $E = E_1 K \supset E_1 \supset E_1^{G(E_1/FK_1)} = FK_1 \supset F$, where the extensions on the ends are by constants and that in the middle is by a group.

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