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On Picard–Vessiot extensions with group PGL_3

Lourdes Juan *, Arne Ledet

Department of Mathematics and Statistics, Texas Tech University, Box 41042, Lubbock, TX 79409-1042, USA

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Abstract

Let F be a differential field of characteristic zero with algebraically closed field of constants. We provide an explicit description of the twisted Lie algebras of PGL_3-equivariant derivations on the coordinate rings of F-irreducible PGL_3-torsors in terms of nine-dimensional central simple algebras over F. We use this to construct a Picard–Vessiot extension which is the function field of a non-trivial torsor and which is a generic extension for PGL_3.

Keywords: Differential Galois theory; Galois cohomology; Generic extensions; Picard–Vessiot

1. Introduction

We work in the standard differential Galois theory context and therefore all the fields considered are of characteristic zero. Let $\mathcal{C}$ be an algebraically closed field with trivial derivation, $G \subset GL_n$ an algebraic group over $\mathcal{C}$ and F a differential field with field of constants $\mathcal{C}$. The following facts are well known:

(1) The Picard–Vessiot extensions (PVEs for short) of F with differential Galois group G, if any, are function fields of F-irreducible G-torsors (cf. [10, Theorem 5.12] or [11, Theorem 1.28]).
(2) The isomorphism classes of G-torsors are in one-to-one correspondence with the equivalence classes of crossed homomorphisms in the first Galois cohomology set $H^1(F, G)$ (cf. [12, Proposition 33]).

* Corresponding author.
E-mail addresses: lourdes.juan@ttu.edu (L. Juan), arne.ledet@ttu.edu (A. Ledet).

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Therefore,

(3) To each PVE with group G one can associate an element of $H^1(F, G)$ which, by Speiser's Theorem [9, Exercise 1.61] corresponds to a matrix $P \in \mathrm{GL}_n(\bar{F})$, $\bar{F}$ denoting the algebraic closure.

The correspondence in (3) can be used to provide a precise description of the 'twisted' Lie algebras of G-equivariant derivations on the function fields of irreducible G-torsors, when a good interpretation of the elements of $H^1(F, G)$ is available—meaning that the matrices P can be given explicitly (see [6] for details).

In the case of PGL_n it is well known that the elements of $H^1(F, \mathrm{PGL}_n)$ correspond to isomorphism classes of n^2-dimensional CSAs (central simple algebras) over F, cf. [9, Section 4.2]. Regrettably, we stop short here since the structure of those is poorly understood in general. In this paper we will discuss the case $n = 3$, where the CSAs are known to be cyclic [1]. We point out that for the case $n = 4$ the structure is also known: the CSAs are crossed products based on a Galois extension for the Klein four group, cf. [1]. However, it is not immediately clear how these can be parametrized, leaving this case open to further research. Finally, the case $n = 2$ is of no interest in this paper since over the fields that we consider PGL_2 is isomorphic to SO_3, a group already studied in [6,7].

Our objective is to provide a systematic description of the PVEs $E \supset F$ with differential Galois group PGL_3, for an arbitrary differential field F with field of constants $\mathcal{C}$, by means of a construction similar to the one done in [7] for the special orthogonal groups. More precisely, let $\mathfrak{gl}_m(\cdot)$ be the Lie algebra of $m \times m$ matrices with coefficients in some specified field, and consider the differential rational field $\mathcal{F} = \mathcal{C}\langle Z_1, \dots, Z_k\rangle$, where $Z_1, \dots, Z_k$ are differentially independent indeterminates over $\mathcal{C}$. We will construct a PVE $\mathcal{E}$ of $\mathcal{F}$ which is the function field of a non-trivial PGL_3-torsor and which satisfies the following:

Definition 1. (See [7].) A Picard–Vessiot G-extension (i.e., a PVE with differential Galois group isomorphic to G) $\mathcal{E} \supset \mathcal{F}$ for the equation $X' = X\mathcal{A}(Z)$, with $\mathcal{A}(Z) = \mathcal{A}(Z_1, \dots, Z_k) \in \mathfrak{gl}_m(\mathcal{F})$ for some m, is said to be *a generic extension for G* if for every Picard–Vessiot G-extension $E \supset F$ there is a specialization $Z_i \to f_i \in F$, such that the equation $X' = X\mathcal{A}(f_1, \dots, f_k)$ gives rise to $E \supset F$ and any fundamental solution matrix maps to one for the specialized equation.

The extension that we construct has the descent generic property [7], meaning that for any differential field F with field of constants $\mathcal{C}$ there is a PVE $E \supset F$ with differential Galois group $H \leqslant G$ if and only if there are $f_i \in F$ such that the matrix $A(f_1, \dots, f_n)$ is well defined, the equation $X' = X\mathcal{A}(f_1, \dots, f_n)$ gives rise to the extension $E \supset F$, and any fundamental solution matrix of $X' = X\mathcal{A}(Z)$ maps to one of $X' = X\mathcal{A}(f_1, \dots, f_n)$ under $Z_i \to f_i$.

Generic extensions (and the related notions of generic equations [2,3]) are broadly discussed in [7]. We point out that the constructions in [4,5] provide a more restricted form of such extensions: in [4] all connected groups are considered, but the extensions are function fields of the trivial torsor. In [5] the groups G considered are the semidirect product $H \ltimes G^0$ of the connected component G^0 by a finite group H, and the extensions are function fields of F-irreducible G-torsors of the form $W \times G^0$, for some F-irreducible H-torsor W. The H-subextensions are also considered fixed. Although these omit the description of extensions which are function fields of non-trivial G-torsors, they cover many more groups than the ones for which an explicit description of the torsors using the first Galois cohomology is known. In [7] we also discuss how the

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extensions in [4] can be used indirectly in the general situation of cohomologically non-trivial connected groups when a good description of the torsors is not available.

2. The representation of PGL_3

Let K be a field. It is well known that an isomorphism $\mathrm{M}_m(K) \otimes_K \mathrm{M}_n(K) \simeq \mathrm{M}_{mn}(K)$ is given by

$$A \otimes 1 \mapsto \begin{pmatrix} A & & & & \\ & A & & & \\ & & A & & \\ & & & \ddots & \\ & & & & A \end{pmatrix}$$

and

$$1 \otimes B \mapsto \begin{pmatrix} b_{11}I & b_{12}I & \dots & b_{1n}I \\ b_{21}I & b_{22}I & \dots & b_{2n}I \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1}I & b_{n2}I & \dots & b_{nn}I \end{pmatrix},$$

or simply

$$(A \otimes B)_{ij} = a_{((i-1) \bmod m)+1,((j-1) \bmod m)+1} \cdot b_{\lfloor (i-1)/m \rfloor+1,\lfloor (j-1)/m \rfloor+1}.$$

Correspondingly, we get a group homomorphism $\mathrm{GL}_n \to \mathrm{SL}_{n^2}$ by

$$A \mapsto A \otimes A^{-T}.$$

Here, the image is closed: If we identify $\mathrm{M}_n(K)$ with K^{n^2}, the image consists exactly of those vector space automorphisms on $\mathrm{M}_n(K)$ that preserve matrix multiplication, i.e., the algebra automorphisms, and this condition clearly defines a Zariski closed set.

Since the kernel consists exactly of the scalar matrices, we have a representation $\varphi : \mathrm{PGL}_n \to \mathrm{SL}_{n^2}$.

We will consider this representation in the case $n = 3$. (And we refrain from writing φ out explicitly.) Now, PGL_3 is an 8-dimensional irreducible algebraic group, and we will need to describe its Lie algebra $\mathfrak{pgl}_3$. As $\mathfrak{pgl}_3$ consists simply of those 9×9 matrices C, for which $I + \varepsilon C$ is in $\mathrm{PGL}_3(K[\varepsilon])$ for an 'algebraic infinitesimal' ε, i.e., a non-zero quantity satisfying $\varepsilon^2 = 0$ (see, e.g., [11, A.2.2]), we can produce such matrices by looking at $(I + \varepsilon A) \otimes (I - \varepsilon A^T)$ for 3×3 matrices A. In fact, if we assume $\mathrm{Tr}(A) = 0$, we get that

$$(I + \varepsilon A) \otimes \left(I - \varepsilon A^T\right) = I + \varepsilon U$$

for

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$$U = A \otimes 1 - 1 \otimes A^T$$
$$= \begin{pmatrix} 0 & a_{12} & a_{13} & -a_{21} & 0 & 0 & -a_{31} & 0 & 0 \\ a_{21} & -a_{11}+a_{22} & a_{23} & 0 & -a_{21} & 0 & 0 & -a_{31} & 0 \\ a_{31} & a_{32} & -a_{22}-2a_{11} & 0 & 0 & -a_{21} & 0 & 0 & -a_{31} \\ -a_{12} & 0 & 0 & a_{11}-a_{22} & a_{12} & a_{13} & -a_{32} & 0 & 0 \\ 0 & -a_{12} & 0 & a_{21} & 0 & a_{23} & 0 & -a_{32} & 0 \\ 0 & 0 & -a_{12} & a_{31} & a_{32} & -a_{11}-2a_{22} & 0 & 0 & -a_{32} \\ -a_{13} & 0 & 0 & -a_{23} & 0 & 0 & a_{22}+2a_{11} & a_{12} & a_{13} \\ 0 & -a_{13} & 0 & 0 & -a_{23} & 0 & a_{21} & a_{11}+2a_{22} & a_{23} \\ 0 & 0 & -a_{13} & 0 & 0 & -a_{23} & a_{31} & a_{32} & 0 \end{pmatrix}. \tag{1}$$

This gives us an 8-dimensional subspace of $\mathfrak{pgl}_3$, i.e., all of $\mathfrak{pgl}_3$.

3. The cohomology

Let K be a field as before. From the short-exact sequence

$$1 \to \bar{K}^* \to \mathrm{GL}_3(\bar{K}) \to \mathrm{PGL}_3(\bar{K}) \to 1,$$

we get (part of) a long-exact cohomology sequence

$$1 \to H^1(K, \mathrm{PGL}_3) \underset{\delta}{\to} H^2(K, \mathbb{G}_m),$$

and if we identify $H^2(K, \mathbb{G}_m)$ with the Brauer group $\mathrm{Br}(K)$, it is known (see, e.g., [9, Lemma 6.3.1]) that

Lemma 1. *For $e \in H^1(K, \mathrm{PGL}_3)$ we have $\delta[e] = [\mathrm{M}_3(K)_e]$, where $\mathrm{M}_3(K)_e$ is the Galois twist of $\mathrm{M}_3(K)$ by e.*

The Galois twists of $\mathrm{M}_3(K)$ are the nine-dimensional central simple algebras over K. These are classically known to be cyclic, cf. [1, Theorem XI.5]. In our case, since the field K contains a primitive third root of unity ζ, this means that the algebras have the form $(a, b/K)_3 = K[i, j]$ for $a, b \in K^*$, where $i^3 = a$, $j^3 = b$ and $ji = \zeta ij$, cf., e.g., [9, 3.5]. These are either *split* (i.e., isomorphic to $\mathrm{M}_3(K)$) or *non-split* (i.e., division algebras).

Thus, we can produce all crossed homomorphisms (and hence all torsors) by starting with a cyclic algebra, and if the cyclic algebra is non-split, the torsor will be non-trivial.

Without loss of generality, we may assume that a is not a third power in K. We then have a C_3-extension $M/K = K(\sqrt[3]{a})/K$ which splits $(a, b/K)_3$, since we can let

$$i = \begin{pmatrix} \sqrt[3]{a} & & \\ & \zeta \cdot \sqrt[3]{a} & \\ & & \zeta^2 \cdot \sqrt[3]{a} \end{pmatrix}, \qquad j = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ b & 0 & 0 \end{pmatrix}.$$

The crossed homomorphism corresponding to $(a, b/K)_3$ then factors through C_3, i.e., it is of the form $e : C_3 \to \mathrm{PGL}_3(M)$. Let $\sigma \in C_3$ be given by $\sigma(\sqrt[3]{a}) = \zeta \cdot \sqrt[3]{a}$. Hence, we can find e

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by finding e_σ, which again can be done by requiring that the matrices i and j above should be invariant under the twisted Galois action of C_3 on $\mathrm{M}_3(M)$, i.e., that

$$e_\sigma \sigma(i) e_\sigma^{-1} = i, \qquad e_\sigma \sigma(j) e_\sigma^{-1} = j.$$

It is obvious that we can simply let $e_\sigma = j^{-1}$. We then have

$$\varphi(e_\sigma) = E_\sigma = \begin{pmatrix} & & & & & & 0 & 0 & 1 \\ & & & & & & b & 0 & 0 \\ & & & & & & 0 & b & 0 \\ 0 & 0 & 1/b & & & & & & \\ 1 & 0 & 0 & & & & & & \\ 0 & 1 & 0 & & & & & & \\ & & & 0 & 0 & 1/b & & & \\ & & & 1 & 0 & 0 & & & \\ & & & 0 & 1 & 0 & & & \end{pmatrix}$$

and get $E_\sigma = P\sigma(P)^{-1}$ for

$$P = \begin{pmatrix} 1 & 0 & 0 & \zeta \cdot \sqrt[3]{a} & 0 & 0 & \zeta^2 \cdot (\sqrt[3]{a})^2 & 0 & 0 \\ 0 & b & 0 & 0 & \zeta \cdot \sqrt[3]{a} b & 0 & 0 & \zeta^2 \cdot (\sqrt[3]{a})^2 b & 0 \\ 0 & 0 & b & 0 & 0 & \zeta \cdot \sqrt[3]{a} b & 0 & 0 & \zeta^2 \cdot (\sqrt[3]{a})^2 b \\ 0 & 0 & 1 & 0 & 0 & \zeta^2 \cdot \sqrt[3]{a} & 0 & 0 & \zeta \cdot (\sqrt[3]{a})^2 \\ 1 & 0 & 0 & \zeta^2 \cdot \sqrt[3]{a} & 0 & 0 & \zeta \cdot (\sqrt[3]{a})^2 & 0 & 0 \\ 0 & b & 0 & 0 & \zeta^2 \cdot \sqrt[3]{a} b & 0 & 0 & \zeta \cdot (\sqrt[3]{a})^2 b & 0 \\ 0 & 1 & 0 & 0 & \sqrt[3]{a} & 0 & 0 & (\sqrt[3]{a})^2 & 0 \\ 0 & 0 & 1 & 0 & 0 & \sqrt[3]{a} & 0 & 0 & (\sqrt[3]{a})^2 \\ 1 & 0 & 0 & \sqrt[3]{a} & 0 & 0 & (\sqrt[3]{a})^2 & 0 & 0 \end{pmatrix}. \tag{2}$$

4. The derivations

We see that

$$P^{-1}P' = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{2b'}{3b} & 0 & 0 & -\frac{\sqrt[3]{a}b'}{3b} & 0 & 0 & -\frac{\sqrt[3]{a^2}b'}{3b} & 0 \\ 0 & 0 & \frac{b'}{3b} & 0 & 0 & \frac{\zeta\cdot\sqrt[3]{a}b'}{3b} & 0 & 0 & \frac{\zeta^2\cdot\sqrt[3]{a^2}b'}{3b} \\ 0 & 0 & 0 & \frac{a'}{3a} & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{b'}{3\sqrt[3]{a}b} & 0 & 0 & \frac{a'}{3a}+\frac{2b'}{3b} & 0 & 0 & -\frac{\sqrt[3]{a}b'}{3b} & 0 \\ 0 & 0 & \frac{\zeta^2 b'}{3\sqrt[3]{a}b} & 0 & 0 & \frac{a'}{3a}+\frac{b'}{3b} & 0 & 0 & \frac{\zeta\cdot\sqrt[3]{a}b'}{3b} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{2a'}{3a} & 0 & 0 \\ 0 & -\frac{b'}{3\sqrt[3]{a^2}b} & 0 & 0 & -\frac{b'}{3\sqrt[3]{a}b} & 0 & 0 & \frac{2a'}{3a}+\frac{2b'}{3b} & 0 \\ 0 & 0 & \frac{\zeta b'}{3\sqrt[3]{a^2}b} & 0 & 0 & \frac{\zeta^2 b'}{3\sqrt[3]{a}b} & 0 & 0 & \frac{2a'}{3a}+\frac{b'}{3b} \end{pmatrix}.$$

This matrix does not have its coefficients in K.

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The derivations on $K[Y, 1/\det Y]$, where $Y = XP$ and X is a generic point for PGL_3, are given by $Y' = YB$, where $B = P^{-1}AP + P^{-1}P'$, and $A \in \mathfrak{pgl}_3(M)$ is picked such that $B \in \mathrm{M}_9(K)$.

We know that there exists an element $C \in P^{-1}\mathfrak{pgl}_3(M)P$ such that $P^{-1}P' - C$ has coefficients in K [6, Proposition 1], meaning that B can then be picked in

$$\begin{aligned}\bigl(P^{-1}\mathfrak{pgl}_3(M)P + P^{-1}P'\bigr)^{C_3} &= \bigl(P^{-1}\mathfrak{pgl}_3(M)P + \bigl(P^{-1}P' - C\bigr)\bigr)^{C_3}\\ &= \bigl(P^{-1}\mathfrak{pgl}_3(M)P\bigr)^{C_3} + \bigl(P^{-1}P' - C\bigr).\end{aligned}$$

Computations show that we can let $C = P^{-1}UP$, where U is the element in $\mathfrak{pgl}_3(M)$ with $a_{11} = -b'/3b$ and the other a_{ij} all 0. Then,

$$P^{-1}P' - C = \mathrm{diag}\bigl(0, \tfrac{2}{3}\tfrac{b'}{b}, \tfrac{1}{3}\tfrac{b'}{b}, \tfrac{1}{3}\tfrac{a'}{a}, \tfrac{1}{3}\tfrac{a'}{a} + \tfrac{2}{3}\tfrac{b'}{b}, \tfrac{1}{3}\tfrac{a'}{a} + \tfrac{1}{3}\tfrac{b'}{b}, \tfrac{2}{3}\tfrac{a'}{a}, \tfrac{2}{3}\tfrac{a'}{a} + \tfrac{2}{3}\tfrac{b'}{b}, \tfrac{2}{3}\tfrac{a'}{a} + \tfrac{1}{3}\tfrac{b'}{b}\bigr).$$

Next, we compute $\mathfrak{T} = (P^{-1}\mathfrak{pgl}_3(M)P)^{C_3}$. This is the twisted Lie algebra associated to the torsor (cf. [6]), and is itself a Lie algebra over K of dimension 8. Finding $\mathfrak{T}$ is just linear algebra, and we find that the elements of $\mathfrak{T}$ are $P^{-1}UP$, where U is an element from $\mathfrak{pgl}_3(M)$ with

$$\begin{aligned}
a_{11} &= x_1 \cdot \sqrt[3]{a} - x_2\zeta(\sqrt[3]{a})^2,\\
a_{12} &= y_1 - y_2 \cdot \sqrt[3]{a} + y_3\zeta^2(\sqrt[3]{a})^2,\\
a_{13} &= z_1 + z_2 \cdot \sqrt[3]{a} + z_3(\sqrt[3]{a})^2,\\
a_{21} &= b\bigl(z_1 + z_2\zeta \cdot \sqrt[3]{a} + z_3\zeta^2(\sqrt[3]{a})^2\bigr) = b\sigma(a_{13}),\\
a_{22} &= x_1\zeta \cdot \sqrt[3]{a} - x_2(\sqrt[3]{a})^2 = \sigma(a_{11}),\\
a_{23} &= y_1 - y_2\zeta \cdot \sqrt[3]{a} + y_3\zeta(\sqrt[3]{a})^2 = \sigma(a_{12}),\\
a_{31} &= b\bigl(y_1 - y_2\zeta^2 \cdot \sqrt[3]{a} + y_3(\sqrt[3]{a})^2\bigr) = b\sigma^2(a_{12}),\\
a_{32} &= b\bigl(z_1 + z_2\zeta^2 \cdot \sqrt[3]{a} + z_3\zeta(\sqrt[3]{a})^2\bigr) = b\sigma^2(a_{13}),
\end{aligned}$$

where $x_1, x_2, y_1, y_2, y_3, z_1, z_2, z_3 \in K$.

For convenience, we scale x_1 and x_2 by $(1+2\zeta)/3$, and y_1, y_2, y_3, z_1, z_2 and z_3 by $(1-\zeta)/3$. The result is

$$\begin{aligned}
&P^{-1}UP\\
&= \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & -az_3 & ax_2 & ay_3 & az_2 & a\zeta x_1 & -ay_2\\
0 & 0 & 0 & -a\zeta y_3 & ab\zeta z_3 & -a\zeta x_2 & a\zeta^2 y_2 & -ab\zeta^2 z_2 & -ax_1\\
0 & -b\zeta^2 y_2 & -bz_2 & 0 & by_1 & -\zeta z_1 & 0 & -aby_3 & -ab\zeta z_3\\
0 & \zeta x_1 & \zeta y_2 & -\zeta z_1 & 0 & y_1 & a\zeta^2 z_3 & ax_2 & 0\\
0 & b\zeta z_2 & -x_1 & y_1 & -b\zeta z_1 & 0 & a\zeta^2 y_3 & 0 & -a\zeta x_2\\
0 & b\zeta y_3 & bz_3 & 0 & by_2 & b\zeta^2 z_2 & 0 & -b\zeta y_1 & bz_1\\
0 & x_2 & -\zeta^2 y_3 & -\zeta z_2 & \zeta x_1 & 0 & z_1 & 0 & -\zeta y_1\\
0 & -b\zeta^2 z_3 & -\zeta x_2 & -\zeta y_2 & 0 & -x_1 & -\zeta y_1 & bz_1 & 0
\end{pmatrix}.
\end{aligned}$$

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Thus, the derivations on $K(Y)$ are given by $Y' = YB$ for

$$B = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{2}{3}\frac{b'}{b} & 0 & -az_3 & ax_2 & ay_3 & az_2 & a\zeta x_1 & -ay_2 \\ 0 & 0 & \frac{1}{3}\frac{b'}{b} & -a\zeta y_3 & ab\zeta z_3 & -a\zeta x_2 & a\zeta^2 y_2 & -ab\zeta^2 z_2 & -ax_1 \\ 0 & -b\zeta^2 y_2 & -bz_2 & \frac{1}{3}\frac{a'}{a} & by_1 & -\zeta z_1 & 0 & -aby_3 & -ab\zeta z_3 \\ 0 & \zeta x_1 & \zeta y_2 & -\zeta z_1 & \frac{1}{3}\frac{a'}{a} + \frac{2}{3}\frac{b'}{b} & y_1 & a\zeta^2 z_3 & ax_2 & 0 \\ 0 & b\zeta z_2 & -x_1 & y_1 & -b\zeta z_1 & \frac{1}{3}\frac{a'}{a} + \frac{1}{3}\frac{b'}{b} & a\zeta^2 y_3 & 0 & -a\zeta x_2 \\ 0 & b\zeta y_3 & bz_3 & 0 & by_2 & b\zeta^2 z_2 & \frac{2}{3}\frac{a'}{a} & -b\zeta y_1 & bz_1 \\ 0 & x_2 & -\zeta^2 y_3 & -\zeta z_2 & \zeta x_1 & 0 & z_1 & \frac{2}{3}\frac{a'}{a} + \frac{2}{3}\frac{b'}{b} & -\zeta y_1 \\ 0 & -b\zeta^2 z_3 & -\zeta x_2 & -\zeta y_2 & 0 & -x_1 & -\zeta y_1 & bz_1 & \frac{2}{3}\frac{a'}{a} + \frac{1}{3}\frac{b'}{b} \end{pmatrix}. \tag{3}$$

5. A Picard–Vessiot extension

Let $a, b, x_1, x_2, y_1, y_2, y_3, z_1, z_2, z_3$ be differentially independent indeterminates over $\mathcal{C}$ and let $\mathcal{K} = \mathcal{C}\langle a_{11}, a_{12}, a_{13}, a_{21}, a_{22}, a_{23}, a_{31}, a_{32}\rangle$ where the a_{ij} are defined in the previous section. Put $Z_{11} = x_1\sqrt[3]{a}$, $Z_{12} = x_2(\sqrt[3]{a})^2$, $Z_{13} = y_1 + \zeta y_3(\sqrt[3]{a})^2$, $Z_{21} = y_1 - (1+\zeta)y_2(\sqrt[3]{a})^2$, $Z_{22} = a_{31}$, $Z_{23} = a_{13}$, $Z_{31} = bz_1 - bz_2\sqrt[3]{a}$, $Z_{32} = bz_1 - bz_3(\sqrt[3]{a})^2$. A calculation shows that $Z_{ij} \in \mathcal{K}$. Since $x_1\sqrt[3]{a}$, $x_2\sqrt[3]{a}^2$, y_1, $y_2(\sqrt[3]{a})^2$, $y_3(\sqrt[3]{a})^2$, z_1, $z_2\sqrt[3]{a}$, $z_3(\sqrt[3]{a})^2$ and b are differentially independent over $\mathcal{C}$ it immediately follows that the Z_{ij} are differentially independent over $\mathcal{C}$. Therefore the differential transcendence degree [8, Definition 3.2.33 and Theorem 5.4.12] of $\mathcal{K}$ over $\mathcal{C}$ has to be eight. This proves that the a_{ij} are differentially independent over $\mathcal{C}$ as well. By [4, Theorem 4.1.2] it follows that $\mathcal{K}(\mathrm{PGL}_3)$ (i.e., the function field of the trivial torsor) is a PVE with group PGL_3 for the equation $X' = XU$, where U is given by (1).

Now, let $\mathcal{F} = \mathcal{C}\langle a, b, x_1, x_2, y_1, y_2, y_3, z_1, z_2, z_3\rangle$. Since $\bar{\mathcal{F}} = \overline{\mathcal{K}\langle a, b\rangle}$ we have that $\bar{\mathcal{F}}(\mathrm{PGL}_3) \supset \mathcal{K}\langle a, b\rangle(\mathrm{PGL}_3)$ is an algebraic and thus no-new-constant extension, since the field of constants of $\mathcal{K}\langle a, b\rangle(\mathrm{PGL}_3)$ is algebraically closed. Thus $\bar{\mathcal{F}}(\mathrm{PGL}_3) \supset \bar{\mathcal{F}}$ is a PVE with group PGL_3. It follows at once that the function field $\mathcal{F}(Y)$ of the PGL_3-torsor corresponding to the matrix P in (2) is a PVE of $\mathcal{F}$ with group PGL_3 for the equation $X' = XB$, with B as in (3). Since a, b are differential indeterminates over $\mathcal{C}$ the associated CSA is non-split and the corresponding torsor is non-trivial.

6. Generic extension

Theorem 1. *The extension $\mathcal{F}(Y) \supset \mathcal{F}$ is a descent generic* PVE *for* PGL_3.

Proof. Let the matrix $\mathcal{A}(Z_i)$ in Definition 1, where the Z_i, $i = 1, \dots, 10$, respectively stand for the differential indeterminates $a, b, x_1, x_2, y_1, y_2, y_3, z_1, z_2, z_3$, be the matrix B in (3).

Suppose that $E \supset F$ is a PVE with differential Galois group $H \leqslant \mathrm{PGL}_3$. Let X, X_H respectively denote generic points of PGL_3 and H. Then $E = F(Y_H)$, where $Y_H = X_H P$ for some invertible matrix P of the form (2), with coefficients in $\bar{F}$. Moreover, there is an F-algebra homomorphism of coordinate rings

$$F\big[XP, \det(XP)^{-1}\big] \twoheadrightarrow F\big[X_H P, \det(X_H P)^{-1}\big].$$

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Since $X_H P$ is a generic point for an H-torsor we have that XP is a generic point for a PGL_3-torsor, and therefore the twisted Lie algebra associated to the H-torsor is contained in that for the PGL_3-torsor. In turn, this implies that the generic point Y_H satisfies an equation with matrix $\tilde{B} = \mathcal{A}(f_i)$ for some $f_i \in F$.

Conversely, a specialization $\mathcal{A}(f_i)$ of $\mathcal{A}(Z_i)$ with $f_i \in F$, gives a derivation on the coordinate ring $F[XP, \det(XP)^{-1}]$ of a PGL_3-torsor, which may have new constants. We get a PVE of F by taking the quotient field of the factor ring

$$F\big[XP, \det(XP)^{-1}\big]/M,$$

where M is a maximal differential ideal. The differential Galois group in this case is the closed subgroup of PGL_3 consisting of those elements that stabilize M.

It is now clear that a fundamental matrix for the equation $\eta' = \eta\mathcal{A}(Z_i)$ specializes to one for $\eta' = \eta\mathcal{A}(f_i)$. For, on the one hand, a solution of $\eta' = \eta\mathcal{A}(Z_i)$ is given by a generic point $XP(Z_1, Z_2)$ of the PGL_3-torsor corresponding to a matrix $P(Z_1, Z_2)$ of the form (2), with a and b, respectively, the differential indeterminates Z_1 and Z_2.

On the other hand, a solution of $\eta' = \eta\mathcal{A}(f_i)$ is given by a generic point $X_H P(f_1, f_2)$ of an H-torsor ($H \leqslant \mathrm{PGL}_3$) corresponding to a matrix $P(f_1, f_2)$ also of the form (2), with a and b, respectively, some elements $f_1, f_2 \in F$.

Clearly, the matrix $P(Z_1, Z_2)$ permits specialization of Z_1 and Z_2 to any non-zero values f_1 and f_2. □

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