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Generic Picard–Vessiot extensions for connected-by-finite groups

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Abstract

We construct generic Picard–Vessiot extensions for linear algebraic groups G which are isomorphic to the semidirect product of a connected group G^0 by a finite group H, where the adjoint H-action on the Lie algebra of G^0 is faithful.

Keywords: Galois theory; Linear differential equations; Picard–Vessiot; Generic extensions

1. Introduction

Generic polynomials with group G have been extensively studied in the context of Galois theory. Work by Noether in connection with a rationality question (see [16] or *the Noether problem* in [6]) already contained such a notion. The precise definition is as follows (cf. [13,21]).

Definition 1.1. Let $\mathbf{s} = (s_1, \ldots, s_m)$ be indeterminates over a field K, and let G be a finite group. A monic polynomial $P(\mathbf{s}, X) \in K(\mathbf{s})[X]$ is called a *generic G-polynomial* over K if the following conditions are satisfied:

(1) The splitting field of $P(\mathbf{s}, X)$ over $K(\mathbf{s})$ is a G-extension, that is, a Galois extension with Galois group isomorphic to G.

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(2) Every G-extension of a field L containing K is the splitting field (over L) of the polynomial $P(\mathbf{a}, X)$ for some $\mathbf{a} = (a_1, \ldots, a_n) \in L^n$. The polynomial $P(\mathbf{a}, X)$ is called a *specialization* of $P(\mathbf{s}, X)$.

The subject has proven relevant not only to the Noether problem but also to the understanding of the structure of G-extensions (cf. [13]). In particular, the above definition is equivalent to Saltman's notion of generic G-extension [19] even though in principle the latter pertains to the more general context of Galois theory of commutative rings [3,13].

A similar notion of *generic linear differential equation with group G* is due to Goldman [4], who produced such equations when G is one of GL_n, SL_n, the reducible group consisting, for fixed r, $1 \leqslant r \leqslant n-1$, of all unimodular matrices $[a_{ij}]$ such that $a_{r+k,m} = 0$, $k = 1, \ldots, n-r$, $m = 1, \ldots, r$, the orthogonal group, or the symplectic group. Bhandari and Sankaran [2] weakened Goldman's conditions and constructed a generic equation for the special orthogonal groups. In [1] Bhandari et al. developed a notion of generic differential modules over differential fields of arbitrary characteristic, but were unable to produce any examples.

In [7,8] the present author constructed Picard–Vessiot G-extensions (i.e., Picard–Vessiot extensions with differential Galois group isomorphic to G) with G connected, that are generic in the sense of Definition 3.1 below. These constructions are restricted to the case when the extensions specialized to are of the form $k(G) \supset k$, where $k(G)$ is the function field of the trivial G-torsor and k is a differential field of characteristic zero with algebraically closed field of constants $\mathcal{C}$.

In this paper we extend the construction in [8] to linear groups $G = H \ltimes G^0$, where H is finite, G^0 is connected and the adjoint H-action on the Lie algebra of G^0 is faithful. Examples of such groups are:

(1) $G = H \ltimes \mathrm{GL}_n$, where H is any finite subgroup of GL_n with no non-identity scalar elements, acting by conjugation.
(2) $G = H \ltimes (\mathbb{G}_a)^n$ where $\mathbb{G}_a$ is the additive group of $\mathcal{C}$, and H is a finite group acting on the vector group $(\mathbb{G}_a)^n$ by matrix multiplication.
(3) $G = A \ltimes (\mathbb{G}_m)^n$, where A is a subgroup of S_n, $\mathbb{G}_m$ denotes the multiplicative group of $\mathcal{C}$, and A acts on $(\mathbb{G}_m)^n$ by permutation of entries.
(4) $G = C_n \ltimes \mathrm{SL}_n$, where C_n acts by conjugation.

We show, constructively, that there are differential fields $\mathcal{E} \supset \mathcal{F}$ such that H acts faithfully on $\mathcal{F}$ as a group of differential automorphisms and $\mathcal{E} \supset \mathcal{F}^H$ is a generic Picard–Vessiot extension with group G. Since this construction uses the one in [8], the results in this paper are restricted to Picard–Vessiot extensions which are the function field $k(V)$ of k-irreducible G-torsors of the form $V = W \times G^0$, where W is a k-irreducible H-torsor. A G-torsor of this form is said to be *split*. The case when $k(V)$ is the function field of a non-split k-irreducible G-torsor remains to be studied.

The present work generalizes the connected case [8] to connected-by-finite groups $H \ltimes G^0$ to the extent it can be done without addressing generic extensions for finite groups. Since the latter have been extensively studied in polynomial Galois theory (see, for example, [3,6,16,18,19,21]) we will not discuss them here.

The rest of the paper is organized as follows: in Section 2 we state without proof some well-known facts about Picard–Vessiot G-extensions and G-torsors. Section 3 summarizes the main steps of the connected case. In Section 4 we recall results by Mitschi–Singer and Hartmann [5,15], that we use to construct a generic Picard–Vessiot extension with group $H \ltimes G^0$. A crite-

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rion for H-equivariance is also established in this section. In Section 5 we discuss what we mean by a generic extension with group $H \ltimes G^0$, and prove the main results concerning this case. Finally, Section 6 illustrates the construction of a generic extension for $G = C_2 \ltimes \mathrm{SL}_2$, with C_2 acting on SL_2 by inverse transposition.

All the fields considered in this paper are of characteristic zero, and $\mathcal{C}$ will denote an algebraically closed field of constants. Unless specified otherwise, G is a linear algebraic group defined over $\mathcal{C}$, and $\mathcal{G}$ is its Lie algebra.

2. A note on Picard–Vessiot extensions and G-torsors

The structure of Picard–Vessiot G-extensions can be described in terms of G-torsors. The following definitions and facts are discussed in [15].

Let G be a linear algebraic group defined over a field k. A *k-homogeneous space* for G is a k-affine variety together with a morphism $G \times V \mapsto V$ of k-varieties inducing a transitive action of $G(\bar{k})$ on $V(\bar{k})$, where $\bar{k}$ denotes the algebraic closure of k. If moreover the action is faithful, V is called a *principal k-homogeneous space* for G or a *G-torsor*. The group G itself is called the trivial G-torsor.

Theorem 2.1. *(Cf. [20,22].) The set of G-torsors (up to G-isomorphism) maps bijectively to the first Galois cohomology set $H^1(k, G)$.*

Theorem 2.2. *(Cf. [10,14,17].) Let k be a differential field with field of constants $\mathcal{C}$ and let $E \supset k$ be a Picard–Vessiot extension with group G. Then E is the function field $k(V)$ of some k-irreducible G-torsor where the action of the Galois group on E is the same as the action resulting from $G(\mathcal{C})$ acting on V. Moreover, $E = k(v)$, for some E-point $v \in V$.*

Note. For a differential field $F \supset \mathcal{C}$ and an F-irreducible G-torsor V, $F(V)$ will denote the function field of V. In particular, if G is connected, $F(G)$ will denote the function field of the trivial G-torsor. Since this notation is very standard, we will often use it without explanation.

3. The connected case

We briefly recall the main results from [8]. In that paper generic extensions which are the function field of the trivial G-torsor were constructed for all connected groups. Those extensions specialize to G-extensions of the same form. Since this restriction to the trivial G-torsor is only apparent in the construction but not explicitly stated, we provide here the following definition in order to make this point clear.

Throughout this section G will be assumed to be connected with Lie algebra $\mathcal{G}$ and $\mathfrak{gl}_m$ will denote the Lie algebra of $m \times m$ matrices.

Definition 3.1. Suppose that $Y_1, \ldots, Y_n$ are differentially independent indeterminates over $\mathcal{C}$, and put $\mathcal{F} = \mathcal{C}\langle Y_1, \ldots, Y_n\rangle$. We say that a Picard–Vessiot G-extension $\mathcal{E} \supset \mathcal{F}$ *is generic for G relative to the trivial G-torsor* if

(1) $\mathcal{E} = \mathcal{F}(G)$, and
(2) for every faithful representation of G in a GL_m, the Picard–Vessiot extension $\mathcal{E} \supset \mathcal{F}$ corresponds to a matrix equation $X' = \mathcal{A}(Y_1, \ldots, Y_n)X$, with $\mathcal{A}(Y_1, \ldots, Y_n) \in \mathcal{G}(\mathcal{F}) \subseteq \mathfrak{gl}_m(\mathcal{F})$

such that, if F is a differential field with field of constants $\mathcal{C}$ and $E \supset F$ is a Picard–Vessiot G-extension with $E = F(G)$, there is a specialization $Y_i \to f_i \in F$, with the equation $X' = \mathcal{A}(f_1, \ldots, f_n)X$ giving rise to $E \supset F$.

Now, put $n = \dim(G)$, let $\mathcal{F} = \mathcal{C}\langle Y_1, \ldots, Y_n\rangle$ as before, and write $\mathcal{R}$ for the coordinate ring $\mathcal{F}[G] = \mathcal{F} \otimes_{\mathcal{C}} \mathcal{C}[G]$ of $G_{\mathcal{F}}$, i.e., the group G seen as a variety over $\mathcal{F}$.

Let $\mathcal{D} = D_{\mathcal{F}} \otimes 1 + \sum_{i=1}^{n} Y_i \otimes D_i$, where $\{D_1, \ldots, D_n\}$ denotes a $\mathcal{C}$-basis of $\mathcal{G}$ and $D_{\mathcal{F}}$ denotes the derivation of $\mathcal{F}$. Then $\mathcal{D}$ is a G-equivariant derivation on $\mathcal{R}$ which extends as such to its quotient field $\mathcal{F}(G)$, the function field of $G_{\mathcal{F}}$. Let $X_1, \ldots, X_n$ be the coordinate functions on G so that $\mathcal{F}(G) = \mathcal{F}(X_1, \ldots, X_n)$. Under the above derivation on $\mathcal{F}(G)$, the X_i become differentially independent over $\mathcal{C}$ and $\mathcal{F}(G) = \mathcal{C}\langle X_1, \ldots, X_n\rangle$. Therefore, the purely differentially transcendental extension $\mathcal{F}(G) \supset \mathcal{C}$ has no new constants and we have (see Theorem 4.1.2 and Corollary 4.1.3 in [8])

Theorem 3.2. *$\mathcal{F}(G) \supset \mathcal{F}$ is a Picard–Vessiot extension with group G.*

Given a faithful representation of G in GL_m, the Lie algebra $\mathcal{G}$ maps to a Lie subalgebra of $\mathfrak{gl}_m$ and the basis $\{D_1, \ldots, D_n\}$ can be identified with a linearly independent set $\{A_1, \ldots, A_n\}$ of $m \times m$ matrices. The matrix $\mathcal{A}(Y_1, \ldots, Y_n) = \sum_{i=1}^{n} Y_i A_i \in \mathcal{G}(\mathcal{F})$ then satisfies the conditions of Definition 3.1.2. By Theorem 4.2.1 in [8], it follows that

Theorem 3.3. *The extension $\mathcal{F}(G) \supset \mathcal{F}$ is a generic Picard–Vessiot extension for G relative to the trivial G-torsor and, furthermore, it descends to subgroups of G as follows*:

Let F be a differential field with field of constants $\mathcal{C}$.

(1) *If $E \supset F$ is a Picard–Vessiot extension with connected differential Galois group $G' \leqslant G$ such that $E = F(G')$, then there is a specialization $Y_i \to f_i \in F$ such that the equation $X' = \mathcal{A}(f_1, \ldots, f_n)X$ gives rise to this extension.*

(2) *For every specialization $Y_i \to f_i \in F$, the differential equation $X' = \mathcal{A}(f_1, \ldots, f_n)X$ gives rise to a Picard–Vessiot extension $E \supset F$ with differential Galois group $G' \leqslant G$.*

4. Preliminaries to the connected-by-finite group case

Let $G = H \ltimes G^0$, where G^0 is connected, H is finite and the adjoint action of H on $\mathcal{G} = \mathrm{Lie}(G^0)$ is faithful. For notational convenience, we identify the connected component of the identity of $H \ltimes G^0$ with its isomorphic image G^0. Likewise we will use H to denote its isomorphic image $(H \ltimes G^0)/G^0$.

Let $\mathcal{F} = \mathcal{C}\langle Y_1, \ldots, Y_n\rangle$, $n = \dim(G)$, be as before. We will show how one can define a faithful action of H on $\mathcal{F}$, as a group of differential automorphisms, and produce a Picard–Vessiot extension $\mathcal{E} \supset \mathcal{F}^H$ with group $H \ltimes G^0$. The main ingredients are a Picard–Vessiot G^0-extension $\mathcal{E} = \mathcal{F}(G^0)$ as in Theorem 3.2, a condition for H-equivariance to be developed next and a criterion by Mitschi–Singer and Hartmann [5,15] to obtain a Picard–Vessiot extension with group G.

For the convenience of the reader, we recall the following material from [15].

Let k be a differential field with field of constants $\mathcal{C}$ and E a Picard–Vessiot extension of k with group G. Let V be as in Theorem 2.2 and write $E = k(v)$ for some E-point $v \in V$. For $\sigma \in G$ and any E-point $r \in V$ denote by ${}^{\sigma}r$ the differential Galois action of σ on r and by $r \cdot \sigma$ the translation action of σ on r via the G-torsor V. We then have ${}^{\sigma}v = v \cdot \sigma$ for all $\sigma \in G(\mathcal{C})$.

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Now suppose that $G = H \ltimes G^0$ and that the G-torsor $V = W \times G^0$, for some k-irreducible H-torsor W. We let F denote the fixed field $E^{G^0} = k(W)$ and write $F = k(w)$ for some F-point $w \in W$ and $E = F(g) = k(g, w)$ for some E-point $g \in G^0$. For $(\sigma, \tau) \in G(\mathcal{C}) = H(\mathcal{C}) \ltimes G^0(\mathcal{C})$ we have

$$^{(\sigma,\tau)}(w, g) = \left(w \cdot \sigma, \sigma^{-1} g \sigma \tau\right),$$

and in particular for $\sigma \in H(\mathcal{C})$

$$^{\sigma} g = \sigma^{-1} g \sigma. \tag{1}$$

Regarding G as a subgroup of some GL_n and its Lie algebra $\mathcal{G}$ as being a subalgebra of the Lie algebra $\mathfrak{gl}_n$ of all $n \times n$ matrices one has that $A = g' g^{-1} \in \mathfrak{gl}_n(F)$, since the entries of A are invariant under the action of the constant group G^0. By (1) we have

$$^{\sigma} A = {}^{\sigma}\left(g' g^{-1}\right) = \left(\sigma^{-1} g \sigma\right)' \sigma^{-1} g^{-1} \sigma = \sigma^{-1} A \sigma. \tag{2}$$

Definition 4.1. (See [5, Definition 3.5], [15, Definition 6.1].) Let K be a Galois extension of k with Galois group H. Let V be a right H-module over k. We consider $K \otimes_k V$ as a left H-module via the action $\sigma \cdot a \otimes v = \sigma(a) \otimes v$ and as a right H-module via the action $a \otimes (v \cdot \sigma)$ for any $\sigma \in H$. We say an element $u \in K \otimes_k V$ is *H-equivariant* if $\sigma \cdot u = u \cdot \sigma$ for all $\sigma \in H$.

With notation as above, consider $V = \mathcal{G}$ as a right H-module via $v \to \sigma^{-1} v \sigma$ for all $\sigma \in H$ and $v \in \mathcal{G}$.

The following proposition from [15] was stated in a context where $k = \mathcal{C}(x)$, but the proof given (see previous discussion for the main points) is also valid when k is a differential field with field of constants $\mathcal{C}$ and E is the function field of a k-irreducible split G-torsor $W \times G^0$, for some k-irreducible H-torsor W. Under these hypotheses we then have

Proposition 4.2. *(See [5, Proposition 3.7], [15, Proposition 6.2].) Let E be a Picard–Vessiot extension of k with Galois group $G = H \ltimes G^0$. Then*

(1) *$K = E^{G^0}$ is the function field of a k-irreducible H-torsor (and so K is a Galois extension of k with Galois group H),*
(2) *E is a Picard–Vessiot extension of K for an equation of the form $X' = AX$ where A is an H-equivariant element of $\mathcal{G}(K)$. Furthermore, the Galois group of E over K is G^0.*

The converse of this result gives a criterion to construct equations with a given Galois group:

Proposition 4.3. *(See [5, Proposition 3.10], [15, Proposition 6.3].) Let k be a differential field with field of constants $\mathcal{C}$. Let $G = H \ltimes G^0 \leqslant \mathrm{GL}_n$ be an algebraic group over $\mathcal{C}$, with H finite and G^0 connected with Lie algebra $\mathcal{G}$. Let W be a k-irreducible H-torsor and let $K = k(W)$.*

Let $A \in \mathcal{G}(K)$ and assume that:

(1) *A is H-equivariant.*
(2) *The Picard–Vessiot extension E of K corresponding to the equation $X' = AX$ has Galois group G^0.*

Then E is the function field of the k-irreducible G-torsor $W \times G^0$ and a Picard–Vessiot extension of k with Galois group G. Furthermore the action of the Galois group corresponds to the action of G on E induced by the action of G on $W \times G^0$.

Moreover, by [15, Proposition 5.1], the condition $A \in \mathcal{G}(K)$ implies that $E = K(g)$ for some $g \in G^0$ with the action of G^0 on g given by right multiplication. The condition that the Galois group is G^0 implies that g is a generic point of G^0.

4.1. A criterion for H-equivariance

As before, let $G = H \ltimes G^0$, where H is finite, G^0 is connected with Lie algebra $\mathcal{G}$, and the adjoint H-action on $\mathcal{G}$ is faithful. Moreover, assume that H acts on G^0 via the right conjugation $h^{-1}gh$, $h \in H$, $g \in G^0$, so that the adjoint action is also a right action. The adjoint H-action is in fact a $\mathcal{C}$-linear action on $\mathcal{G}$ and it induces a faithful representation $\rho : H \to \mathrm{GL}_n(\mathcal{C})$, where $n = \dim(G^0)$, with respect to a basis $\{D_1, \ldots, D_n\}$ of $\mathcal{G}$. Identify H with its isomorphic image $\rho(H) \leqslant \mathrm{GL}_n(\mathcal{C})$. Then for $h \in H$, the H-action on the D_i reads:

$$D_i \cdot h = \sum_{j=1}^{n} \rho(h)_{ij} D_j \tag{3}$$

where the $\rho(h)_{ij}$ are the entries of the matrix $\rho(h)$.

Now, let $M = \sum_{i=1}^{n} \mathcal{C} m_i$ be the $\mathcal{C}$-span of n elements m_i in some field extension of $\mathcal{C}$. Assume that H and the m_i (not necessarily linearly independent over $\mathcal{C}$) are such that there is a left H-action on M. For example, take $m_1 = x \notin \mathcal{C}$, $m_2 = -x$ and $H = C_2 = \{h_1, h_2\}$. Let $\rho : H \to \mathrm{GL}_2$ be the representation given by

$$\rho(h_1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \rho(h_2) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

then

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} m_1 \\ m_2 \end{pmatrix} = \begin{pmatrix} m_2 \\ m_1 \end{pmatrix}.$$

So, there is a well defined action on $\sum_{i=1}^{2} \mathcal{C} m_i$ by letting $h_1 \cdot m_i = m_i$, $h_2 \cdot m_1 = m_2$ and $h_2 \cdot m_2 = m_1$.

The extended Lie algebra $M \otimes_{\mathcal{C}} \mathcal{G}$ can then be seen as a left H-module via the action $h \cdot m \otimes D$ and as a right H-module via the action $m \otimes D \cdot h$.

Proposition 4.4. *Let $\mathcal{D} = \sum_{i=1}^{n} m_i \otimes D_i \in M \otimes_{\mathcal{C}} \mathcal{G}$. Then, for $h \in H$, $h \cdot \mathcal{D} = \mathcal{D} \cdot h$ if and only if the H-action on the m_i is given by*

$$h \cdot m_i = \sum_{j=1}^{n} \rho(h)^T_{ij} m_j, \tag{4}$$

where $\rho(h)^T$ denotes the transpose of the matrix $\rho(h)$.

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Proof. The condition $h \cdot \mathcal{D} = \mathcal{D} \cdot h$ reads $\sum_{i=1}^{n}(h \cdot m_i) \otimes D_i = \sum_{i=1}^{n} m_i \otimes (D_i \cdot h)$. Substituting the expression in (3) for $D_i \cdot h$, this last equation implies

$$\begin{aligned}\sum_{i=1}^{n}(h \cdot m_i) \otimes D_i &= \sum_{i=1}^{n} m_i \otimes \left\{\sum_{j=1}^{n} \rho(h)_{ij} D_j\right\} \\ &= \sum_{j=1}^{n}\left\{\sum_{i=1}^{n} \rho(h)_{ij} m_i\right\} \otimes D_j \\ &= \sum_{k=1}^{n}\left\{\sum_{\ell=1}^{n} \rho(h)^T_{k\ell} m_\ell\right\} \otimes D_k,\end{aligned}$$

that is,

$$\sum_{i=1}^{n}\left\{(h \cdot m_i) - \sum_{j=1}^{n} \rho(h)^T_{ij} m_j\right\} \otimes D_i = 0. \tag{5}$$

By [12, Proposition 2.3, Chapter XVI] it then follows that

$$h \cdot m_i = \sum_{j=1}^{n} \rho(h)^T_{ij} m_j. \qquad \square$$

The following is an immediate consequence:

Corollary 4.5. *Let $m_1, \ldots, m_n$, $D_1, \ldots, D_n$ be as above and suppose that the m_i are algebraically* (*respectively differentially*) *independent over $\mathcal{C}$. A left H-action may then be defined in the field $\mathcal{C}(m_1, \ldots, m_n)$* (*respectively differential field $\mathcal{C}\langle m_1, \ldots, m_n\rangle$*) *via Eq.* (4)*, such that the element $\mathcal{D} = \sum_{i=1}^{n} m_i \otimes D_i$ satisfies $h \cdot \mathcal{D} = \mathcal{D} \cdot h$.*

Remarks. Suppose we start with a left H-action on G and make it a right action in the standard way, namely, via $h^{-1} \cdot g$, $h \in H$, $g \in G$. In this case, if $\rho : H \to \mathrm{GL}_n$ denotes the representation induced by the adjoint left action, with respect to the same basis, then the dual representation to ρ given by $\rho^\vee(h) = (\rho(h)^{-1})^T$, $h \in H$, produces the result.

The H-equivariance of $\mathcal{D}$ is related to the fact that for a k-vector space V with basis $\{v_i\}$ and dual basis $\{v_i^\vee\}$ under the isomorphism $E^\vee \otimes E \cong \mathrm{End}_k(E)$ [12, p. 628] the Casimir element $\sum_{i=1}^{n} v_i^\vee \otimes v_i$ is sent to the identity.

5. The connected-by-finite case

Fix a linear algebraic group $G = H \ltimes G^0$ as before and let $\mathcal{G}$ be its Lie algebra. Note that $\mathcal{G} = \mathrm{Lie}(G^0)$.

Definition 5.1. Let F be a differential field containing $\mathcal{C}$ on which H acts faithfully as differential automorphisms. Regard $\mathcal{G}$ as a Lie subalgebra of the Lie algebra of $m \times m$ matrices $\mathfrak{gl}_m$ for some m, and let $A \in \mathcal{G}(F)$. We say that the equation $X' = AX$ is *H-equivariant* if the matrix A is H-equivariant in the sense of Definition 4.1.

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We generalize Definition 3.1 as follows.

Definition 5.2. Suppose that $Y_1, \dots, Y_n$ are differentially independent indeterminates over $\mathcal{C}$ and put $\mathcal{F} = \mathcal{C}\langle Y_1, \dots, Y_n\rangle$. We say that a Picard–Vessiot G-extension $\mathcal{E} \supset \mathcal{K}$ is *generic for G relative to split G-torsors* if there is a faithful differential H-action on $\mathcal{F}$, with $\mathcal{K} = \mathcal{F}^H$, such that

(1) $\mathcal{E} = \mathcal{K}(\mathcal{W} \times G^0)$ for some $\mathcal{K}$-irreducible H-torsor $\mathcal{W}$, and
(2) for every faithful representation of G in a GL_m, the G^0-extension $\mathcal{F}(G^0) \supset \mathcal{F}$ has an H-equivariant equation $X' = \mathcal{A}(Y_1, \dots, Y_n)X$, such that, given a Picard–Vessiot G-extension of the form $k(W \times G^0) \supset k$, where k is a differential field with field of constants $\mathcal{C}$ and W a k-irreducible H-torsor, there is an H-equivariant specialization $Y_i \to f_i$ with $f_i \in k(W)$, such that the G^0-extension $k(W)(G^0) \supset k(W)$ has equation $X' = \mathcal{A}(f_1, \dots, f_n)X$.

Remarks 5.3.

(1) Note that if a matrix $A(Y_1, \dots, Y_n) \in \mathcal{G}(\mathcal{F})$ is H-equivariant then a specialization $A(f_1, \dots, f_n)$ will be automatically H-equivariant. This is trivially verified by observing that, under the specialization $Y_i \to f_i$, $h^{-1}A(Y_1, \dots, Y_n)h = A(Y_1, \dots, Y_n) \to h^{-1}A(f_1, \dots, f_n)h = A(f_1, \dots, f_n)$, $h \in H$.
(2) It is well known that if k is a differential field and $F \supset k$ is a Galois extension with group H then the derivation of k extends uniquely to F making $F \supset k$ a Picard–Vessiot extension and H a group of differential automorphisms. Note, however, that if F is a differential field on which H acts faithfully, but not necessarily differentially, the extension $F \supset F^H$ is Galois, but not necessarily differential: let $F = \mathcal{C}(x_1, x_2)$ be a rational field with derivation given by $D(x_1) = 1$, $D(x_2) = x_2$, and consider the C_2-action on F given by $x_1 \to x_2$, $x_2 \to x_1$. Obviously, this is not a differential action. It is not hard to see that the fixed field F^{C_2} is not a differential field either (consider the element x_1x_2). Thus the hypothesis that the H-action on $\mathcal{F}$ is both faithful and differential is necessary.

We now show how to construct the extension $\mathcal{E} \supset \mathcal{K}$. The first step is the following

Proposition 5.4. *Let H be a finite group that acts as a group of automorphisms of a connected linear algebraic group G^0 such that the adjoint action of H on the Lie algebra $\mathcal{G} = \mathrm{Lie}(G^0)$ is faithful. Let $\mathcal{F} = \mathcal{C}\langle Y_1, \dots, Y_n\rangle$, where the Y_i are differentially independent indeterminates over $\mathcal{C}$ and $n = \dim(G^0)$. There exist*

(1) *a faithful differential action of H on $\mathcal{F}$, and*
(2) *an element $\mathcal{D}(Y_1, \dots, Y_n) \in \mathcal{G}(\mathcal{F})$ that is H-equivariant*

such that if F is any differential field with field of constants $\mathcal{C}$ on which H acts faithfully as differential automorphisms and D is any H-equivariant element of $\mathcal{G}(F)$, then $D = \mathcal{D}(f_1, \dots, f_n)$ for some choice of $f_i \in F$ and furthermore, the specialization $Y_i \to f_i$ is H-equivariant.

Proof. Let $\rho : H \to \mathrm{GL}_n(\mathcal{C})$ represent the adjoint H-action on $\mathcal{G}$ with respect to a basis $\{D_1, \dots, D_n\}$. Since the Y_i are differentially independent, by Corollary 4.5, we can define a left H-action on $\mathcal{F}$ such that the element $\mathcal{D}(Y_1, \dots, Y_n) = \sum_{i=1}^n Y_i \otimes D_i \in \mathcal{F} \otimes_{\mathcal{C}} \mathcal{G}$ satisfies

$h \cdot \mathcal{D}(Y_1, \ldots, Y_n) = \mathcal{D}(Y_1, \ldots, Y_n) \cdot h$, where $h \cdot \mathcal{D}(Y_1, \ldots, Y_n)$ denotes the action of H on the $\mathcal{F}$ factor of $\mathcal{F} \otimes_{\mathcal{C}} \mathcal{G}$ and $\mathcal{D}(Y_1, \ldots, Y_n) \cdot h$, the action on the $\mathcal{G}$ factor.

The action on $\mathcal{F}$ is given by an action on the Y_i via $h \cdot Y_i = \sum_{j=1}^n \rho(h)_{ij}^T Y_j$, which extends differentially to the derivatives $Y_i^{(k)}$ in an obvious way, that is, $h \cdot Y_i^{(k)} = \sum_{j=1}^n \rho(h)_{ij}^T Y_j^{(k)}$—note that the coefficients $\rho(h)_{ij}^T$ are constant.

Since the H-action on $\mathcal{F}$ is both faithful and differential and, moreover, $\mathcal{F}^H \supset \mathcal{C}$ it then follows that $\mathcal{F} \supset \mathcal{F}^H$ is a Picard–Vessiot extension with group H. Thus, as an element of the left and right H-module $\mathcal{F} \otimes_{\mathcal{F}^H} (\mathcal{F}^H \otimes_{\mathcal{C}} \mathcal{G})$, $\mathcal{D}(Y_1, \ldots, Y_n)$ is also H-equivariant in the sense of Definition 4.1.

Let F and $D \in \mathcal{G}(F)$ be as in the hypotheses and write $D = \sum_{i=1}^n f_i D_i$, $f_i \in F$. Obviously, $D = \mathcal{D}(f_1, \ldots, f_n)$. Since D is H-equivariant, by Proposition 4.4, the H-action on the f_i is given by (4). The latter implies that the specialization $Y_i \to f_i$ is H-equivariant, that is, $h \cdot Y_i = \sum_{j=1}^n \rho(h)_{ij}^T Y_j \to \sum_{j=1}^n \rho(h)_{ij}^T f_j = h \cdot f_i$. □

Note that for the sake of the last proposition, only the fact that the extension $\mathcal{F} \supset \mathcal{F}^H$ is Galois was needed (see Definition 4.1). The fact that this extension is also differential will be used in what follows.

Let $\mathcal{F}$ and D_i be as above. We identify G with a subgroup of GL_m for some m, and $\mathcal{G}$ with a Lie subalgebra of the Lie algebra of $m \times m$ matrices $\mathfrak{gl}_m$. The D_i then map to a matrix basis A_i of $\mathcal{G}$ and the element $\mathcal{D} = \sum_{i=1}^n Y_i \otimes D_i$, to an H-equivariant matrix

$$A(Y_1, \ldots, Y_n) = \sum_{i=1}^n Y_i A_i \in \mathcal{G}(\mathcal{F}). \tag{6}$$

Since $\mathcal{F} \supset \mathcal{F}^H$ is a Picard–Vessiot extension with group H, there is an $\mathcal{F}^H$-irreducible H-torsor $\mathcal{W}$ such that $\mathcal{F} = \mathcal{F}^H(\mathcal{W})$. By Theorem 3.2, the field $\mathcal{E} = \mathcal{F}(G^0)$ is a Picard–Vessiot extension of $\mathcal{F}$ with Galois group G^0 for the equation $X' = A(Y_1, \ldots, Y_n)X$. Since $A(Y_1, \ldots, Y_n)$ is H-equivariant, by Proposition 4.3 we have

Proposition 5.5. *$\mathcal{E}$ is the function field of the $\mathcal{F}^H$-irreducible G-torsor $\mathcal{W} \times G^0$ and a Picard–Vessiot extension of $\mathcal{F}^H$ with Galois group G.*

Let k be a differential field with field of constants $\mathcal{C}$ and W a k-irreducible H-torsor. Assume that $E = k(W \times G^0) \supset k$ is a Picard–Vessiot extension with group G and let $F = k(W) = E^{G^0}$. By Proposition 4.2, there is an H-equivariant matrix $A \in \mathcal{G}(F)$ such that the G^0-extension $E = F(G^0) \supset F$ has equation $X' = AX$. Proposition 5.4 then implies that there is an H-equivariant specialization $Y_i \to f_i$, with $f_i \in F$, such that $A = A(f_1, \ldots, f_n) = \sum_{i=1}^n f_i A_i$. This shows

Theorem 5.6. *$\mathcal{E} \supset \mathcal{F}^H$ is a generic Picard–Vessiot extension for G relative to split G-torsors.*

In polynomial Galois theory a generic polynomial is said to be *descent generic* if 1.1.2, can be replaced with the stronger condition:

(2′) For any subgroup $H \leqslant G$, every H-extension of a field L containing K is the splitting field (over L) of the polynomial $P(\mathbf{a}, X)$ for some $\mathbf{a} = (a_1, \ldots, a_n) \in L^n$.

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Kemper [9] has shown that generic polynomials are always descent generic.

Goldman [4] and Bhandari–Sankaran [2] had a similar requirement as a part of their definitions of generic equation. But the existence of generic differential equations satisfying such restriction has only been verified to date for the groups in [2,4].

In our case, however, we can show that the descent theorem for connected groups (Theorem 3.3), can also be extended to this situation. For this we first prove the following generalization of Proposition 4.3.

Proposition 5.7. *Let $H, G' \leqslant \mathrm{GL}_m$ be algebraic groups over $\mathcal{C}$, with H finite and G' not necessarily connected. Le F be a differential field with field of constants $\mathcal{C}$ on which H acts faithfully as a group of differential automorphisms with $\mathcal{C} \subset F^H$. Let W be an F^H-irreducible H-torsor such that $F = F^H(W)$.*

Let $A \in \mathfrak{gl}_m(F)$ and assume that:

(1) *A is H-equivariant.*
(2) *The Picard–Vessiot extension E of F corresponding to the equation $X' = AX$ has Galois group G'.*

Then there is a conjugation action of H on G' such that E is the function field of an F^H-irreducible $H \ltimes G'$-torsor $W \times V$, and a Picard–Vessiot extension of F^H with Galois group $H \ltimes G'$. Furthermore the action of the Galois group corresponds to the action of $H \ltimes G'$ on E induced by the action of $H \ltimes G'$ on $W \times V$.

Proof. We follow the proof of Proposition 6.3 in [15] making changes when necessary. First notice that since H acts faithfully and differentially on F, $F \supset F^H$ is a Picard–Vessiot extension with group H since $\mathcal{C} \subset F^H$ by hypothesis.

By [15, Lemma 3.1], there is a matrix $w \in \mathrm{GL}_m(F)$ such that for any $\sigma \in H \subset \mathrm{GL}_m(\mathcal{C})$ we have that ${}^\sigma w = w\sigma$. It is then clear that the action of H on $F^H(w)$ is faithful and therefore the Galois group of the extension $F \supset F^H(w)$ is trivial. Thus, $F = F^H(w)$. By Theorem 2.2 there is an F-irreducible G'-torsor V such that $E = F(v)$ for some E-point $v \in V$. Note that since the coordinate rings $\overline{F}[V]$ and $\overline{F}[G']$, where $\overline{F}$ denotes the algebraic closure, are isomorphic, without loss of generality, we may assume that v is an $m \times m$ matrix. It then follows that $E = F^H(w, wv)$.

By assumption the constant subfield of E is $\mathcal{C}$ and $Y = \mathrm{diag}(w, wv) \in \mathrm{GL}_{2m}(E)$ satisfies $Y' = \overline{A}Y$ where

$$\overline{A} = \begin{pmatrix} w'w^{-1} & 0 \\ 0 & w'w^{-1} + wAw^{-1} \end{pmatrix}.$$

Clearly $w'w^{-1}$ and $w'w^{-1} + wAw^{-1}$ have coefficients in F. Since A is H-equivariant, both of these are invariant under the action of H and they must lie in F^H. Therefore, E is a Picard–Vessiot extension of F^H. Since $\mathrm{Gal}(E/F) = G'$ and $\mathrm{Gal}(F/F^H) = H$ we have an exact sequence of groups:

$$(1) \to G' \to \mathrm{Gal}\bigl(E/F^H\bigr) \to H \to (1).$$

The differential action of H on F can be extended to E via $h \cdot v = h^{-1}vh$. In fact, $h^{-1}v'h = h^{-1}Avh = h^{-1}Ahh^{-1}vh = Ah^{-1}vh$, since A is H-equivariant. Since the action of H on F is faithful, this gives an injective homomorphism from H to $\mathrm{Gal}(E/F^H)$. The embedding of H in $\mathrm{Gal}(E/F^H)$ is induced by its action on the fundamental solution matrix $Y = \mathrm{diag}(w, wv)$. Since $h \cdot Y = (wh, wvh)$ the image of $h \in H$ in $\mathrm{Gal}(E/F^H)$ is $\mathrm{diag}(h, h)$. The image of G' in $\mathrm{Gal}(E/F^H)$ is $\mathrm{diag}(I, G')$. Therefore $\mathrm{Gal}(E/F^H)$ is isomorphic to $H \ltimes G'$. □

As before, we fix a representation of G in GL_m for some m and regard its Lie algebra $\mathcal{G}$ as a subalgebra of the Lie algebra of $m \times m$ matrices $\mathfrak{gl}_m$. We keep the notations $Y_1, \ldots, Y_n$ for differentially independent indeterminates over $\mathcal{C}$, $\mathcal{F} = \mathcal{C}\langle Y_1, \ldots, Y_n\rangle$, and $A(Y_1, \ldots, Y_n) = \sum_{i=1}^n Y_i A_i \in \mathcal{G}(\mathcal{F})$ as in (6).

Theorem 5.8. *Let $\mathcal{E} \supset \mathcal{F}^H$ be the generic Picard–Vessiot G-extension relative to split G-torsors from Theorem* 5.6. *Let F be a differential field with field of constants $\mathcal{C}$ on which H acts faithfully as differential automorphisms with $\mathcal{C} \subset F^H$.*

1. *Assume that $G' \leqslant G^0$ is a connected H-stable subgroup. If $E \supset F$ is a Picard–Vessiot G'-extension with $E = F(G')$, there is an H-equivariant specialization $Y_i \to f_i \in F$ such that $F(G') \supset F$ has equation $X' = A(f_1, \ldots, f_n)X$.*
2. *Let $Y_i \to f_i \in F$ be an H-equivariant specialization such that the equation $X' = A(f_1, \ldots, f_n)X$ has Picard–Vessiot extension $E \supset F$ with* (*not necessarily connected*) *group $G' \leqslant \mathrm{GL}_m$. Then G' is an H-stable subgroup of G^0 and $E \supset F^H$ is a Picard–Vessiot extension with group $H \ltimes G' \leqslant G$.*

Proof. Note that under the present hypotheses we have that $\mathcal{F}(G^0) \supset \mathcal{F}$ is a generic Picard–Vessiot extension for G^0 relative to the trivial G^0-torsor.

To prove 1, suppose that $G' \leqslant G^0$ is a connected H-stable subgroup. If $E \supset F$ is a Picard–Vessiot G'-extension with $E = F(G')$, by Theorem 3.3.1, there is a specialization $Y_i \to f_i$, with $f_i \in F$, such that $\mathcal{A}(f_1, \ldots, f_n) \in \mathcal{G}(F)$ and $F(G') \supset F$ has equation of the form $X' = \mathcal{A}(f_1, \ldots, f_n)X$. By Remark 5.3.1, $\mathcal{A}(f_1, \ldots, f_n)$ is H-equivariant. Repeating the argument used in the proof of Proposition 5.4, one can show that the specialization $Y_i \to f_i$ is also H-equivariant.

For 2, let $Y_i \to f_i \in F$ be an H-equivariant specialization such that $X' = A(f_1, \ldots, f_n)X$ has Picard–Vessiot extension with group G'. Then, by Theorem 3.3.2, $G' \leqslant G^0$. Since $A = A(f_1, \ldots, f_n)$ is an H-equivariant element of $\mathcal{G}(F)$, by Proposition 5.7, it then follows that $E \supset F^H$ is a Picard–Vessiot extension with group $H \ltimes G'$.

Notice that since the conjugation action of the matrix group H on both G' and G^0 is via matrix multiplication we furthermore have that G' is an H-stable subgroup of G^0 and $H \ltimes G' \leqslant H \ltimes G^0$. □

6. Example

Let $G^0 = \mathrm{SL}_2(\mathcal{C})$ and $H = C_2$, where the action of C_2 on $\mathrm{SL}_2(\mathcal{C})$ is given by inverse transposition, that is, conjugation by the element $\left(\begin{smallmatrix} 0 & -i \\ i & 0 \end{smallmatrix}\right)$. Applying the adjoint action to the basis

$$A_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad A_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

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of Lie(SL_2), we see that this action is diagonalizable with eigenvectors A_1, A_2, A_3. The corresponding eigenvalues are 1, −1, −1.

Let σ denote the nontrivial element of C_2 and $\rho : C_2 \to GL_3(\mathcal{C})$ be the representation given by the above action on the basis $\{A_1, A_2, A_3\}$. Then,

$$\rho(\sigma) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Observe that in this case $\rho^T(\sigma) = \rho(\sigma)$. Following the method above, we let C_2 act on the differentially independent elements Y_i, $i = 1, \ldots, 3$, by $\sigma \cdot Y_1 = Y_1$, $\sigma \cdot Y_2 = -Y_2$, $\sigma \cdot Y_3 = -Y_3$. Then $\mathcal{C}\langle Y_1, Y_2, Y_3\rangle(SL_2) \supset \mathcal{C}\langle Y_1, Y_2, Y_3\rangle^{C_2}$ is a generic Picard–Vessiot extension for $C_2 \ltimes SL_2$. The equation corresponding to the standard representation of SL_2 in GL_2 is

$$X' = \mathcal{A}(Y_1, Y_2, Y_3)X, \quad \text{where } \mathcal{A}(Y_1, Y_2, Y_3) = \begin{pmatrix} Y_3 & -Y_1 + Y_2 \\ Y_1 + Y_2 & -Y_3 \end{pmatrix}.$$

Now, let $k = \mathcal{C}(x)$, $x' = 1$, and F be the quadratic extension $\mathcal{C}(\sqrt{x}\,)$. Then the C_2-equivariant specialization $Y_1 \to 1$, $Y_2 \to \sqrt{x}$, $Y_3 \to -\sqrt{x}$ gives

$$A = A\big(1, \sqrt{x}, -\sqrt{x}\,\big) = \begin{pmatrix} -\sqrt{x} & -1 + \sqrt{x} \\ 1 + \sqrt{x} & \sqrt{x} \end{pmatrix}$$

which has group SL_2 over F. The following proof of this fact was provided by Jacques-Arthur Weil. It uses an implementation of Kovacic's Algorithm [11] in Maple.

Choose as cyclic vector $V := [1, I]$, with $I^2 = -1$. Consider the system the system $Z' = AZ$, $Z = [Z_1, Z_2]^T$ and let $f := 1 \cdot Z_1 + I \cdot Z_2 = V \cdot Z$ (scalar product). Then our system is equivalent to an operator that has coefficients in $\mathcal{C}(x)$ (i.e., the operator *descends* to $\mathcal{C}(x)$). The operator satisfied by f is

$$L := \partial^2 - \frac{\partial}{2x} - \frac{-I - 2x + 4x^2}{2x}.$$

Applying Kovacic's Algorithm to L one sees that there are no Liouvillian solutions. Therefore, the Galois group of L is $C_2 \ltimes SL_2$. The Maple code used in this calculation is:

```
with(DEtools): _Envdiffopdomain:=[Dx,x];
A:=matrix(2,2,[-sqrt(x),-1+sqrt(x),1+sqrt(x),sqrt(x)]);
B,P:=cyclic(A,[1,I]);
L:=Dx^2-B[2,2]*Dx-B[2,1]; eq:=diffop2de(L,y(x)):
kovacicsols(eq,y(x));
```

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