Differential Projective Modules over Differential Rings, II

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ABSTRACT. Differential modules over a commutative differential ring R which are finitely generated projective as ring modules, with differential homomorphisms, form an additive category, so their isomorphism classes form a monoid. We study the quotient monoid of this monoid by the submonoid of isomorphism classes of free modules with component wise derivation. This quotient monoid has the reduced K_0 of R (ignoring the derivation) as an image and contains the reduced K_0 of the constants of R as its subgroup of units. This provides a description of the isomorphism classes of differential projective R modules up to an equivalence.

1. Introduction

Let R be a commutative differential ring with derivation D_R, or just D if the context is clear. A *differential projective module* over R is a finitely generated projective R module P and an additive endomorphism D_M, or just D if the context is clear, such that $D_M(rm) = D_R(r)m + rD_M(m)$ for all $r \in R$ and $m \in M$. For example, if $A \in M_n(R)$ defining D on the free module $R^{(n)}$ by $D((x_1, \ldots, x_n) = (x'_1, \ldots, x'_n) + (x_1, \ldots, x_n)A$ makes the free module into a differential projective module denoted $P(A)$ or $(R^{(n)}, A)$ whose differential structure is denoted D_A. When A is a zero matrix, we call the differetial module *trivial.* In [**4**] we investigated differential projective modules and looked at their classification. It is our goal in this work to advance this classification. In [**4**], we took a K theoretic approach. We review briefly that approach and its difficulties.

The isomorphism classes of differential projective modules, with an addition operation induced from direct sum, form a commutative monoid with identity the class of the zero module; the most general group to which this monoid maps is denoted $K_0^{\text{diff}}(R)$ [**4**, p. 43434]. We use brackets to denote the image of a differential projective module in $K_0^{\text{diff}}(R)$. For differential projective modules M and N, $[M] = [N]$ provided there is a differential projective module P such that $M \oplus P$ is differentially isomorphic to $N \oplus P$. There is a differential projective module Q such that $P \oplus Q$ is differentially isomorphic to $P(C)$ for suitable C [**4**, Corollary 3, p. 4342] so we conclude that $[M] = [N]$ provided there is a C with $M \oplus P(C)$ differentially isomorphic to $N \oplus P(C)$.

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Forgetting the differential structure defines a homomorphism from this differential K group to the usual K group, $K_0^{\text{diff}}(R) \to K_0(R)$. This is in epimorphism by [**4**, Theorem 2, p. 4341], and every element of the kernel has the form $[P(C)] - [P(D)]$ where C and D are matrices of the same size by [**4**, p. 4242]. Informally, these results say that, setting aside the non–differential K-theory of R, the differential K theory of R is controlled by the classes of differential projective modules whose underlying R module is free. Thus the classification problem, using K theory, comes down to the relation of declaring $P(A)$ and $P(B)$ equivalent provided there is a C with $P(A) \oplus P(C)$ differentially isomorphic to $P(B) \oplus P(C)$.

Differential projective modules $P(A)$ and $P(B)$ are differentially isomorphic if there is an invertible matrix T with $T' = AT - TB$. Thus $P(A) \oplus P(C)$ is differentially isomorphic to $P(B) \oplus P(C)$ provided there is an invertible matrix S with $S' = (A \oplus C)S - S(B \oplus C)$. To tell if $P(A)$ and $P(B)$ are equivalent requires considering all possible C's and S's for which this equation holds. Although this is just a problem of matrix algebra, controlling the complications arising from stabilization with C are formidable.

To avoid these complications, we propose a finer classification, where we restrict the C's to be zero matrices. (Example 1 shows that this classification can actually be finer.) This leads us to a quotient of the monoid of isomorphism classes which is a monoid, but need not be a group.

Starting with the monoid of isomorphism classes of differential projective modules we look at the quotient by the submonoid of isomorphism classes of modules of the form $P(0_n)$. While this quotient differential monoid is not always a group, it maps surjectively to the similar object defined for R ignoring the derivation, which is a group, and which we show is isomorphic to $K_0(R)/ < [R] >$. When R is connected, this quotient is the kernel of the rank function. Moreover, we show that the group of units of the quotient differential monoid is isomorphic to $K_0(R^D)/ < [R^D] >$. Also, we will see that the part of the differential quotient monoid not described by the K theory of R and R^D is captured by the quotient of the submonoid of differential isomophism classes of the modules of the form $P(A)$. (This approach is similar to the one adopted for differential Azumaya algebras in [**5**], an approach taken because of the difficulties similar to those noted above that appear in [**3**]).)

2. The Monoids

A monoid is a commutative semigroup with identity. We use addition for the operation, and 0 for the identity. Except for the notation for the operation, the following observations are identical with [**5**]: Throughout this section we will be concerned with monoids of isomorphism classes and various quotients thereof. If $N \subseteq M$ is a submonoid, by M/N we mean the set of equivalence classes on M under the equivalence relation $m_1 \sim m_2$ if there are $n_1, n_2 \in N$ such that $m_1 + n_1 = m_2 + n_2$. Sums of equivalent elements are equivalent, which defines a commutative operation on M/N by adding representatives of equivalence classes, and the equivalence class of the identity is an identity. Thus M/N is a monoid. The projection $p : M \to M/N$ ($p(m)$ is the class of m) is an epimorphism, $p(N) = 0$, and any monoid homomorphism $q : M \to Q$ with $q(N) = 0$ factors uniquely through p. Note, however, that we can have $M/N = 0$ but $N \neq M$, for example $\mathbb{Q}^*/\mathbb{Z}^\times$. Thus $p^{-1}(0)$ can be strictly larger than N.

If N is a subgroup of the monoid M, M/N is the set of N orbits in M where N acts by translation.

The set of invertible elements $U(M)$ of M is its maximal subgroup. The set of elements $a \in M$ whose equivalence classes in M/N are invertible is $\{a \in M | \exists b \in M, n \in N \text{ such that } a+b+n \in N\}$. The monoid $M/U(M)$ has no units except the identity.

We will be considering the following monoids. In the definitions we sometimes consider R as a commutative ring and sometimes as a differential ring, and similarly for differential projective R modules.

Definition 1. *$MP(R)$ denotes the set of isomorphism classes of finitely generated projective R modules with the operation induced by direct sum and the identity the isomorphism class of* 0.

$MP^{diff}(R)$ denotes denotes the set of isomorphism classes of differential finitely generated projective R modules with the operation induced by direct sum and the identity the isomorphism class of 0.

$MF(R)$ denotes the submonoid of $MP(R)$ whose elements are isomorphism classes of free R modules.

$MF^{diff}(R)$ denotes the submonoid of $MP^{diff}(R)$ whose elements are isomorphism classes of differential projective modules of the form $(R^{(n)}, Z)$.

$MF_0^{diff}(R)$ denotes the submonoid of $MP^{diff}(R)$ whose elements are isomorphism classes of differential projective modules of the form $(R^{(n)}, 0)$.

$PCM(R)$ and $\tilde{K}_0(R)$ both denote $MP(R)/MF(R)$

$PCM^{diff}(R)$ denotes $MP^{diff}(R)/MF_0^{diff}(R)$

$FCM^{diff}(R)$ denotes $MF^{diff}(R)/MF_0^{diff}(R)$

$PCM(R)$ is called the projective class monoid *of R.*

$PCM^{diff}(R)$ is called the differential projective class monoid *of R.*

$FCM^{diff}(R)$ is called the diffferential free module class monoid *of R.*

All of the monoids in Definition 1 are functors.

Using Definiton 1 and the definition of quotient monoids we see that if P and Q are finitely generated projective R modules (respectively differential finitely generated projective R modules) then their classes in $PCM(R)$ (respectively $PCM^{\text{diff}}(R)$) will be equal provided for some m and n $P \oplus R^{(n)}$ is isomorphic to $Q \oplus R^{(m)}$ (respectively $P \oplus (R^{(n)}, 0)$ is differentially isomorphic to $Q \oplus (R^{(m)}, 0)$). In particular, P is zero in $PCM(R)$ (respectively $PCM^{\text{diff}}(R)$) provided for some m and n $P \oplus R^{(n)}$ is isomorphic to $R^{(m)}$ (respectively $P \oplus (R^{(n)}, 0)$ is differentially isomorphic to $(R^{(m)}, 0)$). In this case P is called stably free (respectively *stably differentially trivial.*

Proposition 1. *$PCM(R)$ is a group isomorphic to $K_0(R)/ < [R] >$. If R is connected, $PCM(R)$ is isomorphic to the kernel of the rank homomorphism $K_0(R) \to \mathbb{Z}$.*

Proof. Let P be a finitely generated projective R module. The existence of a Q such that $P \oplus Q$ is a free module shows that the class of Q is an inverse to the class of P in $MP(R)/MF(R)$. Thus $PCM(R)$ is a group. Hence the map $MP(R) \to PCM(R)$ factors through $K_0(R)$, and since $[R] \mapsto 0$ it factors through $K_0(R)/ < [R] >$. On the other hand, the monoid map $MP(R) \to K_0(R)$ maps $MF(R)$ into $< [R] >$ so that $MP(R) \to K_0(R)/ < [R] >$ factors through

$MP(R)/MF(R)$. The composites in either order are the identity, so $PCM(R)$ is isomorphic to $K_0(R)/ < [R] >$. Suppose R is connected, so that the rank homomorphism $rk : K_0(R) \to \mathbb{Z}$ is defined. Then $< [R] > \cap \mathrm{Ker}(rk) = 0$ and $\mathrm{Ker}(rk) + < [R] >= K_0(R)$ so that $K_0(R)/ < [R[>$ is isomorphic to $\mathrm{Ker}(rk)$. □

The kernel of rk is denoted $Rk_0(R)$ in [**2**, p. 459]. Sometimes the kernel of rk is called the *reduced* K group and denoted $\tilde{K}_0(R)$; we have made the definition so that the latter always denotes $PCM(R)$ whether R is connected or not.

The homomorphism $MP^{\mathrm{diff}}(R) \to MP(R)$ by dropping the derivation is surjective [**4**, Theorem 2, p. 4341], so the induced map $PCM^{\mathrm{diff}}(R) \to PCM(R)$ is an epimorphism. This map carries $FCM^{\mathrm{diff}}(R)$ to 0. In fact we have the following:

Proposition 2. *The map $PCM^{diff}(R) \to PCM(R)$ is an epimorphism. The map $p : PCM^{diff}(R)/FCM^{diff}(R) \to PCM(R)$ is an isomorphism*

Proof. As we have remarked, [**4**, Theorem 2, p. 4341] implies the first assertion and so p is an epimorphism. To see that it is an isomorphism, we first show that it is a homomorphism of groups and then show it has trivial kernel. To see that $PCM^{\mathrm{diff}}(R)/FCM^{\mathrm{diff}}(R)$ is a group, let P be a differential finitely generated projective R module. By [**4**, Corollary 3 p. 4342] there is a differential projective module Q such that $P \oplus Q$ is of the form $P(A)$ for some A. That is, the class of P plus the class of Q in $MP^{\mathrm{diff}}(R)$ lies in $MF^{\mathrm{diff}}(R)$. Thus $MP^{\mathrm{diff}}(R)/MF^{\mathrm{diff}}(R)$ is a group, hence so is its image $PCM^{\mathrm{diff}}(R)/FCM^{\mathrm{diff}}(R)$. Now suppose again that P is a differential finitely generated projective module such that its class in $PCM^{\mathrm{diff}}(R)/FCM^{\mathrm{diff}}(R)$ is sent to 0 by p. Thus as projective R modules P and 0 have the same class in $PCM(R)$, so there are m and n such that $P \oplus R^{(n)}$ is isomorphic to $R^{(m)}$. Thus, using the given differential structure on P, $P \oplus (R^{(n)}, 0)$ is a differential module free (of rank m) as an R module and hence of the form $(R^{(m)}, Z)$. It follows that the class of P in $PCM^{\mathrm{diff}}(R)/FCM^{\mathrm{diff}}(R)$ is trivial. Thus p has trivial kernel, so is injective, and hence an isomorphism. □

Informally, Proposition 2 can be understood as asserting that the classification of differential projective R modules comes down to the classification of those which are free as R modules, since when the latter are nullified the differential structures on projective modules only depend on the R module structure. More formally:

Corollary 1. *If $FCM^{diff}(R) = 0$ then $PCM^{diff}(R) = PCM(R)$.*

Corollary 2. *If $PCM(R) = 0$ then $FCM^{diff}(R) = PCM^{diff}(R)$.*

Proof. (Of Corollary 2.) As remarked above, it is possible for the quotient of a monoid by a proper submonoid to not be trivial. We argue as in the proof of Proposition 2. Since $PCM(R) = 0$, for any projective R module P there are m,n such that $P \oplus R^{(n)}$ is isomorphic to $R^{(m)}$. Now suppose P has a differential structure. Then so does $P \oplus (R^{(n)}, 0)$, which is then isomorphic to $(R^{(m)}, A)$ for some A. Since the class of $(R^{(m)}, A)$ is in $FCM^{\mathrm{diff}}(R)$, so is the class of $P \oplus (R^{(n)}, 0)$, and this is the same as the class of P. □

Proposition 2 is also, in a way, a statement about the functor from differential modules to modules. A counterpart to that is the functor from modules over the constants to differential modules, which results in the following:

PROPOSITION 3. *The functor $P_0 \mapsto R \otimes_{R^D} P_0$ induces a group isomorphism $q : PCM(R^D) \to U(PCM^{diff}(R))$.*

PROOF. The functor induces a map $MP(R^D) \to MP^{\text{diff}}(R)$. If P_0 is a free R^D module of finite rank, then $R \otimes_{R^D} P_0$ belongs to $MF_0^{\text{diff}}(R)$, so the functor passes to a homomorphism $PCM(R^D) \to PCM^{\text{diff}}(R)$. If P_0 is a finitely generated projective R^D module, and Q_0 is a finitely generated projective R^D module such that $P_0 \oplus Q_0$ is free, then in $PCM^{\text{diff}}(R)$ the classes $R \otimes_{R^D} P_0$ and $R \otimes_{R^D} Q_0$ add up to the class of 0, and hence are inverses of each other. Thus the homomorphism $PCM(R^D) \to PCM^{\text{diff}}(R)$ has image in the units of the range. Suppose P is a differential finitely generated projective module whose class in $PCM^{\text{diff}}(R)$ is invertible. Then there exist m, n, and a differential finitely generated projective module Q such that $P \oplus Q \oplus (R^{(n)}, 0)$ is isomorphic to $(R^{(m)}, 0)$. In particular, P is a differential direct summand of $(R^{(m)}, 0)$. By [**4**, Theorem 1, p. 4340], this means that P is of the form $R \otimes_{R^D} P_0$ where P_0 is a finitely generated projective R^D module. This shows that q is group epimorphism. Suppose that the class of P_0 is in the kernel of q. Then the classes of $R \otimes_{R^D} P_0$ and 0 are the same in $PCM^{\text{diff}}(R)$ so there are m, n such that $(R \otimes_{R^D} P_0) \oplus (R^{(n)}, 0)$ is differentially isomorphic to $(R^{(m)}, 0)$. Also $(R \otimes_{R^D} P_0) \oplus (R^{(n)}, 0)$ is differentially isomorphic to $R \otimes_{R^D} (P_0 \oplus (R^D)^{(n)})$ and $(R^{(m)}, 0)$ is differentially isomorphic to $R \otimes_{R^D} (R^D)^{(m)}$. Taking constants and applying [**4**, Lemma 1 p. 4340] we see that $P_0 \oplus (R^D)^{(n)}$ is R^D isomorphic to $(R^D)^{(m)}$. That is, P_0 and 0 are the same class in $PCM(R^D)$. Thus q has trivial kernel and hence is an isomorphism. □

The functor $P_0 \mapsto R \otimes_{R^D} P_0$ also induces a group homomorphism $PCM(R^D) \to PCM(R)$. This need not be either injective or surjective, in general. However Proposition 3 enables the identification of its kernel.

COROLLARY 3. *The kernel of $\phi : PCM(R^D) \to PCM(R)$ is, under the isomorphism q of Proposition 3, $U(FCM^{diff}(R))$.*

PROOF. The inclusion $R^D \to R$ induces the group homomorphism $PCM(R^D) \to PCM(R)$. The kernel of this homomorphism is given by classes of finitely generated projective R^D modules P_0 such that $P = R \otimes_{R^D} P_0$ is a stably free R module: there are m, n such that $P \oplus R^{(n)} = R \otimes_{R^D} (P_0 \oplus (R^D)^{(n)})$ is isomorphic to $R^{(m)}$. Since P_0 and $P_0 \oplus (R^D)^{(n)}$ give rise to the same class in $PCM(R^D)$, we can replace the former by the latter and assume that P is actually R free. So the class of P as a differential module lies in $FCM^{\text{diff}}(R)$. Since it comes from $PCM(R^D)$ it also lies in $U(PCM^{\text{diff}}(R))$ and hence in $FCM^{\text{diff}}(R) \cap U(PCM^{\text{diff}}(R))$. Thus q of the kernel lies in this intersection; since under the morphism $PCM^{\text{diff}}(R) \to PCM(R)$, $FCM^{\text{diff}}(R)$ goes to zero, it is isomorphic to the kernel under q. We claim this intersection is $U(FCM^{\text{diff}}(R))$, which will complete the proof. That $U(FCM^{\text{diff}}(R))$ is contained in the intersection is clear. On the other hand the intersection, being the kernel of a group homomorphism, is a (sub)group in $FCM^{\text{diff}}(R)$ and hence conatined in its group of units. □

Using Proposition 3 we have the following consequences of Corollaries 1 and 2:

COROLLARY 4. *If $FCM^{diff}(R) = 0$ then $PCM(R^D) \to PCM(R)$ is an isomorphism.*

Proof. Suppose $FCM^{\text{diff}}(R) = 0$. By Corollary 1, $PCM^{\text{diff}}(R) = PCM(R)$. By Proposition 1, $PCM(R)$ is a group, hence $PCM^{\text{diff}}(R) = U(PCM^{\text{diff}}(R)$, so by Proposition 3 $PCM(R^D)$ maps isomorphically to $PCM^{\text{diff}}(R)$ and hence $PCM(R^D) \to PCM(R)$ is an isomorphism. □

Corollary 5. *If $PCM(R^D) = 0$ then $PCM^{\text{diff}}(R)$ has no invertible elements.*

Proof. Combine Corollary 2 and Proposition 3. □

Proposition 2 implies that if $PCM^{\text{diff}}(R) = 0$ then $PCM(R) = 0$. Of course if $PCM^{\text{diff}}(R) = 0$ then so does its submonoid $PCM^{\text{diff}}(R)$. Then Corollary 4 implies that $PCM(R^D) = PCM(R) = 0$. We record this fact, along with the similar assertion under the stronger assumption that all differential projective R modules are trivial:

Corollary 6. *If $PCM^{\text{diff}}(R) = 0$ then $PCM(R^D) = PCM(R) = 0$. If all finitely generated differential projective R modules are trivial, then all finitely generated projective R modules and all finitely generated projective R^D modules are free.*

Proof. For the second assertion, let P be a finitely generated projective R module. Let D_P be a differential structure on P. Since P with D_P is trivial, its underlying module P is free. Let Q be a finitely generated projective R^D module. Since $R \otimes_{R^D} Q$ is trivial, it is differentially isomorphic to $(R, 0)^{(n)}$ for some n. It follows that $Q = (R \otimes_{R^D} Q)^D = ((R, 0)^{(n)})^D = (R^D)^{(n)}$. □

In Proposition 5 below we will see a partial converse of Corollary 6

We have recalled the fact that every finitely generated projective R module carries a differential structure. Choosing such a structure for each finitely generated projective R module amounts to a map from the set of finitely generated projective R modules to the set of differential projective R modules which is a right inverse to the forgetful function from the latter to the former. Suppose the choices could be made so that: isomorphic modules had equivalent differential structures; the structure on a direct sum was the direct sum of the structures on each summand; and the structure on R was $(R, 0)$. Informally, we would say we have a canonical choice of differential structure. Or in other words, suppose there were a right inverse ψ to the epimorphism $PCM^{\text{diff}}(R) \to PCM(R)$. Since (by Proposition 1) $PCM(R)$ is a group, so is $\psi(PCM(R))$, which is isomorphic to it. Now suppose that R is such that $PCM(R^D) = 0$ and $PCM(R) \neq 0$. Then by Corollary 5 $PCM^{\text{diff}}(R)$ has no invertible elements, so its subgroup $\psi(PCM(R))$ is trivial. This contradiction means that for such a ring R there is no canonical choice for differential structures on projective modules. For a concrete example of such an R we can take the Picard–Vessiot ring R of a Picard–Vessiot extension of a differential field F with differential Galois group G such that $F[G]$ has non stably free projective modules. Then $R^D = F$ so $PCM(R^D) = 0$ while $PCM(R) = PCM(F[G]) \neq 0$.

3. Trivial Differential Projective Class Monoid

We consider the condition that $PCM^{\text{diff}}(R) = 0$, or that all projective differential modules are stably trivial. We are going to see that this condition is quite restrictive, much more so than the corresponding condition that all projectives are stably free.

If $PCM^{\text{diff}}(R) = 0$, then the submonoid $FCM^{\text{diff}}(R)$ is also trivial. A partial converse also holds:

PROPOSITION 4. *If $PCM^{diff}(R) = 0$ then $FCM^{diff}(R) = 0$. If $FCM^{diff}(R) = 0$ and $PCM(R^D) = 0$ then $PCM^{diff}(R) = 0$.*

PROOF. We have already noted the first implication. So suppose $FCM^{\text{diff}}(R) = 0$ and let P be a finitely generated differential projective module. Then there is a differential projective Q such that $P \oplus Q$ is free as an R module which, as an element of $FCM^{\text{diff}}(R)$ must be 0. Thus the class of P is invertible in $PCM^{\text{diff}}(R)$. Proposition 3 and the assumption that $PCM(R^D) = 0$ imply that the class of P is 0. □

The second implication of Proposition 4 says that if all differential R modules which are free as R modules are stably trivial, and if all projective R^D modules are stably free, then all differential projective R modules are stably trivial. This compares to the corresponding non–differential condition that all projective modules are stably free. The stronger condition, namely that all projective modules are free, has often been investigated. The differential analogue would be the condition that all differential protectives are trivial. As it happens, the differential and non–differential conditions are related:

PROPOSITION 5. *If all finitely generated projective R^D modules are free, then every stably trivial finitely generated differential projective R module P is trivial.*

PROOF. The hypothesis is that there are m, n such that $P \oplus (R^{(m)}, 0)$ is isomorphic to $(R^{(n)}, 0)$. In particular, P is a differential direct summand of a trivial differential module. As remarked several times previously, this implies that P is of the form $R \otimes_{R^D} P_0$ for some finitely generated projective R^D module P_0. Since by assumption P_0 is free, say for rank k, P differentially isomorphic to $(R^{(k)}, 0)$ and hence trivial. (Of course $k = m - n$.) □

If $\eta : R \to S$ is a differential ring homomorphism, then there is an induced monoid homomorphism $h : PCM^{\text{diff}}(R) \to PCM^{\text{diff}}(S)$ which carries the submonoid $FCM^{\text{diff}}(R)$ of $PCM^{\text{diff}}(R)$ to the submonoid $FCM^{\text{diff}}(S)$ of $PCM^{\text{diff}}(S)$. If η is surjective, h need not be, but the restriction of h to $FCM^{\text{diff}}(R)$ is. If it happens that $PCM(S^D) = 0$, then we can use this to see when $PCM^{\text{diff}}(R) \neq 0$.

COROLLARY 7. *Let $\eta : R \to S$ be a surjective differential homomorphism of differential rings. Assume $PCM(S^D) = 0$. If $PCM^{diff}(R) = 0$ then $PCM^{diff}(S) = 0$.*

PROOF. If $PCM^{\text{diff}}(R) = 0$, then by Proposition 4 $FCM^{\text{diff}}(R) = 0$. Since η is surjective, this implies that $FCM^{\text{diff}}(S) = 0$. Then by Proposition 4 again, $PCM^{\text{diff}}(S) = 0$. □

If S is a simple differential ring, then S^D is a field, so all projective S modules are free, and in particular $PCM(S^D) = 0$. So if $R \to S$ is a surjection with S simple, and if $PCM^{\text{diff}}(R) = 0$, then $PCM^{\text{diff}}(S) = 0$.

COROLLARY 8. *Let R be a differential ring with $PCM^{diff}(R) = 0$. Then for every maximal differential ideal I of R, $PCM^{diff}(R/I) = 0$.*

Note that in general R/I, for I maximal differential, may have differential projective modules which are not obtained from differential projective R modules by reduction modulo I. Indeed every differential R module, projective or not, reduces to a differential projective R/I module [**1**].

We consider as examples a class of differential rings whose differential projective class monoids are *never* trivial.

We will be using some facts about Picard–Vessiot extensions, which we now briefly recall ([**7**, Chapter 1]). Let F be a characteristic zero differential field with algebraically closed field of constants C. Let $P = (F^{(n)}, A)$ be a finitely generated differential projective F module. A differential field extension $E \supseteq F$ is called a *Picard–Vessiot extension* of F for P provided: (1) $E^D = C$; (2) $E \otimes_F P$ is trivial; and (3) No proper differential subfield of E containing F satisfies (1) and (2). Picard–Vessiot extensions exist and are unique up to isomorphism. Now suppose P_1 and P_2 are finitely generated projective differential F modules which represent the same class in $PCM^{\text{diff}}(F)$. That is, there are m, n such that $P_1 \oplus (R, 0)^{(m)}$ and $P_2 \oplus (R, 0)^|(n)$ are isomorphic. Let E_i be a Picard–Vessiot extension for E_i. Since $E_1 \otimes_F P_1$ has a basis of constants, so does $E_1 \otimes_F (P_1 \oplus (R, 0)^{(m)})$. Because of the isomorphism, $E_1 \otimes_F (P_2 \oplus (R, 0)^{(n)})$ also has a basis of constants. In other words, $E_1 \otimes_F (P_2 \oplus (R, 0)^{(n)})$ is trivial. Thus $E_1 \otimes_F P_2$ is stably trivial, and then by Proposition 5 $E_1 \otimes_F P_2$ is trivial, i.e. has a basis of constants. Since also $E_1^D = C$, E_1 contains a Picard–Vessiot extension for P_2. Similarly, E_2 contains a Picard–Vessiot extension of P_1. We conclude that E_1 and E_2 are isomorphic differential F algebras. If we choose all Picard–Vessiot extensions of F insode the same Picard–Vessiot closure of F, then we even have $E_1 = E_2$. Conversely, if the Picard–Vessiot extensions for P_1 and P_2 are not equal, then P_1 and P_2 have different classes in $PCM^{\text{diff}}(F)$. This same reasoning applies to sets of differential F modules with more than two members, indeed to sets of any cardinality:

PROPOSITION 6. *Let F be a characteristic zero differential field with algebraically closed field of constants C. Let $\{P_i | i \in I\}$ be a set of finitely generated differential projective F modules and for each $i \in I$ let E_i be a Picard–Vessiot extension of F for P_i, all inside a chosen Picard–Vessiot closure of F. Suppose all the E_i's are distinct. Then the classes of the P_i's are distinct elements of $PCM^{\text{diff}}(F)$.*

Next we look at affine differential algebras over F:

PROPOSITION 7. *Let F be a characteristic zero differential field with algebraically closed field of constants C. Suppose that F has Picard–Vessiot extensions of arbitrarily large transcendence degree over F. (This is a relatively mild condition satisfied for $F = \mathbb{C}$ and $F = \mathbb{C}(x)$.) Let R be a differential R algebra which is finitely generated as an F algebra. Then $PCM^{\text{diff}}(R) \neq 0$.*

PROOF. By Corollary 8 it is enough to show that for some maximal differential Ideal I of R $PCM^{\text{diff}}(R/I) \neq 0$. Thus we can replace R by R/I and assume that R is differentially simple. It follows that the quotient field E of R is an extension of F whose field of constants is also C [**6**]. Note that E has finite transcendence degree over F, so there are finitely generated differential modules over F which do not have Picard–Vesiot extensions in E. In particular, this means there is some matrix A over F such that $P(A) = (F^{(n)}, A)$ does not have a basis of constants over E, and hence does not have one over R. Thus $P(A)$ is not a trivial differential R module. Since R^D is the field C, all projective R^D modules are free. Proposition 5 then implies that $P(A)$ is not even stably trivial, and hence its class in $PCM^{\text{diff}}(R)$ is non–trivial. □

As noted in the statement of Proposition 7, the result applies to differential affine $\mathbb{C}$ algebras, none of which, including polynomial rings or Laurent polynomial rings, can have trivial differential projective class monoids. We consider finally a specific algebra of this latter type, which illustrates the distinction between equivalence in the monoid and equivalence in the corresponding K group.

Example 1. *Let* $S = \mathbb{C}[a, b, c, u, u^{-1}, v, v^{-1}]$ *be the differential ring where* $D(\alpha) = 0$ *for* $\alpha \in \mathbb{C}$, $D(a) = D(b) = D(c) = 0$, $D(u) = (a-c)u$ *and* $D(v) = (b-c)v$. *Let* A *be the diagonal matrix with entries* a, c *and let* B *be the diagonal matrix with entries* b, c. *We will see that* $P(A) = (S, a) \oplus (S, c)$ *is isomorphic to* $P(B) = (S, b) \oplus (S, c)$ *but that there is no* n *such that* $(S, a) \oplus (S, 0)^{(n)}$ *is isomorphic to* $(S, a) \oplus (S, 0)^{(n)}$.

Let T_0 be the matrix

$$\begin{bmatrix} 0 & u \\ v & 1 \end{bmatrix}$$

Then T_0 is invertible, and a calculation shows that $T_0' = AT_0 - T_0B$. Thus T_0 gives a differential isomorphism from $P(A)$ to $P(B)$. Now suppose we have an invertible $(n+1) \times (n+1)$ matrix $T = [t_{ij}]$ giving an isomorphism from $(S, a) \oplus (S, 0)^{(n)} = (S^{(n+1)}, \text{diag}(a, 0, \ldots, 0))$ to $(S, a) \oplus (S, 0)^{(n)} = (S^{(n+1)}, \text{diag}(b, 0, \ldots, 0))$. Then $T' = \text{diag}(a, 0, \ldots, 0)T - T\text{diag}(b, 0, \ldots, 0)$ so $t'_{11} = (a-b)t_{11}$, $t'_{1i} = at_{1i}$ and $t'_{i1} = -bt_{i1}$ for $i > 1$, and otherwise $t'_{ij} = 0$. Thus we need to calculate derivatives in S. Let $R = \mathbb{C}[a, b, c]$ so that $S = R[u, v][(uv)^{-1}]$. Then $t \in S$ can be written $t = (\sum_{p,q} a_{pq}u^pv^q)(uv)^{-l}$ where $a_{pq} \in R$ and p, q, l are non–negative integers. Then $t' = (\sum_{p,q} a_{pq}(p(a-c) + q(b-c) - l((a-c) + (b-c))u^pv^q)(uv)^{-l}$. Suppose we have $t' = (a-b)t$. Comparing terms shows that for all p, q $a_{pq}(a-b) = a_{pq}(p(a-c) + q(b-c) - l(a-c+b-c)$. If $a_{pq} \neq 0$ then $a - b = (p-l)a + (q-l)b + (2l-p-q)c$ so $p - l = q - l = 1$ and $2l - p - q = 0$. These conditions imply that $l = 1$ and $p = q = 0$. Thus $t = a_{00}(uv)^{-1}$. This then implies that $t' = -(a-c+b-c)t$ and $(a-b)t \neq (a-c+b-c)t$ unless $t = 0$. Thus in the matrix T $t_{11} = 0$. Similar calculations show that if $t' = at$ then $t = 0$ so that in the matrix T $t_{1i} = 0$ for $i > 1$. Thus the whole first column of T consists of zeros. Thus there is no invertible matrix T giving a differential isomorphism. Thus (S, a) and (S, c) are not equivalent in $PCM^{\text{diff}}(S)$. Since $(S, a) \oplus (S, c)$ is isomorphic to $(S, b) \oplus (S, c)$, (S, a) and (S, b) are equivalent in $K_0^{\text{diff}}(S)$.

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