Math 4351, Fall 2018
Chapter 11 in Goldberg

1 Measurable Sets

Our goal is to define what is meant by a measurable set $E \subseteq [a, b] \subset \mathbb{R}$ and a measurable function $f : [a, b] \to \mathbb{R}$. We defined the length of an open set and a closed set, denoted as $|G|$ and $|F|$, e.g., $|[a, b]| = b - a$. We will use another notation for complement and the notation in the Goldberg text. Let $E^c = [a, b] \setminus E = [a, b] - E$. Also,

$$E_1 \setminus E_2 = E_1 - E_2.$$

Definitions: Let $E \subseteq [a, b]$. **Outer measure** of a set E: $\overline{m}(E) = \inf\{|G| : \text{for all } G \text{ open and } E \subseteq G\}$. **Inner measure** of a set E: $\underline{m}(E) = \sup\{|F| : \text{ for all } F \text{ closed and } F \subseteq E\}$. $0 \leq \underline{m}(E) \leq \overline{m}(E)$. A set E is a **measurable set** if $\overline{m}(E) = \underline{m}(E)$ and the measure of E is denoted as $m(E)$. The **symmetric difference** of two sets E_1 and E_2 is defined as

$$E_1 \Delta E_2 = (E_1 - E_2) \cup (E_2 - E_1).$$

A set is called an F_σ set **(F-sigma set)** if it is a union of a countable number of closed sets. A set is called a G_δ set **(G-delta set)** if it is a countable intersection of open sets.

Properties of Measurable Sets on $[a, b]$:

1. If E_1 and E_2 are subsets of $[a, b]$ and $E_1 \subseteq E_2$, then $\underline{m}(E_1) \leq \underline{m}(E_2)$ and $\overline{m}(E_1) \leq \overline{m}(E_2)$. In addition, if E_1 and E_2 are measurable subsets of $[a, b]$ and $E_1 \subseteq E_2$, then $m(E_1) \leq m(E_2)$.

2. The empty set is a measurable set and $m(\emptyset) = 0$.

3. All open sets $G \subseteq [a, b]$ and all closed sets $F \subseteq [a, b]$ are measurable sets and $m(G) = |G|$, $m(F) = |F|$.

4. If E is a measurable set, then the complement of E in $[a, b]$ is a measurable set and $m(E) + m(E^c) = b - a$.

5. The Cantor set $C \subset [0, 1]$ is a measurable set and $m(C) = 0$.

6. If E_1 and E_2 are measurable sets, then $E_1 \cup E_2$ and $E_1 \cap E_2$ are measurable sets and

$$m(E_1) + m(E_2) = m(E_1 \cup E_2) + m(E_1 \cap E_2).$$

7. If E_1 and E_2 are measurable sets, then $E_1 - E_2$ is a measurable set.

8. If $\{E_n\}_{n=1}^{\infty}$ is a countable collection of measurable sets, then the union $\cup_{n=1}^{\infty} E_n$ and the intersection $\cap_{n=1}^{\infty} E_n$ are measurable sets. Also,

$$m\left(\bigcup_{n=1}^{\infty} E_n\right) \leq \sum_{n=1}^{\infty} m(E_n)$$

9. If the countable collection $\{E_n\}_{n=1}^{\infty}$ of measurable sets is pairwise disjoint, $E_j \cap E_k = \emptyset$, $j \neq k$, then

$$m\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} m(E_n).$$

10. Not all subsets of $[a, b]$ are measurable. That is, there exists subsets $E \subset [a, b]$ that are NOT measurable.

The following theorem shows that sets of measure zero do not affect the measurability of a set. (See 11.3 I.) The preceding 10 properties are used to prove the following Theorem and the In-Class Problems.

Theorem 1. *Let $E_1, E_2 \subseteq [a,b]$. If E_1 is measurable and the symmetric difference $m(E_1 \Delta E_2) = 0$, then E_2 is measurable and $m(E_2) = m(E_1)$.*

Proof. The set E_2 can be expressed as follows (draw a picture):

$$E_2 = \underbrace{[E_1 \cup (E_2 - E_1)]}_{E_1 \cup E_2} - [E_1 - E_2]. \tag{1.1}$$

The symmetric difference $m(E_1 \Delta E_2) = 0$ and any subset of a set of measure zero is measurable and has measure zero (Exercise 11.2#3). Therefore, $E_1 - E_2$ and $E_2 - E_1$ are measurable and $m(E_1 - E_2) = 0$ and $m(E_2 - E_1) = 0$. The union of two measurable sets is measurable; $E_1 \cup (E_2 - E_1)$ is measurable. Since both $E_1 \cup (E_2 - E_1)$ and $E_2 - E_1$ are measurable, it follows from Equation (1.1) and property 7 that E_2 is measurable. Now, we need to show that $m(E_2) = m(E_1)$.

Note that $E_1 \cup E_2$ can be written as the union of two disjoint measurable sets, either E_1 and $E_2 - E_1$ or E_2 and $E_1 - E_2$. Therefore,

$$\begin{aligned} m(E_1 \cup E_2) = m(E_1 \cup (E_2 - E_1)) &= m(E_1) + m(E_2 - E_1) = m(E_1) \\ m(E_1 \cup E_2) = m(E_2 \cup (E_1 - E_2)) &= m(E_2) + m(E_1 - E_2) = m(E_2). \end{aligned}$$

Therefore, $m(E_2) = m(E_1)$. □

In Class Problems:

1. True or False? If G is an open subset of $[a,b]$ and $|G| = 0$, then $G = \emptyset$.

2. True or False? If F is a closed subset of $[a,b]$ and $|F| = 0$, then $F = \emptyset$.

3. Prove: If $E \subset [a,b]$ and $\overline{m}(E) = 0$, then E is measurable and $m(E) = 0$.

4. Give examples of an F_σ set that is not closed and a G_δ set that is not open.

5. Prove: Every subset of $[a,b]$ that is of type F_σ is measurable.

6. Prove: Every subset of $[a,b]$ that is of type G_δ is measurable.

7. True or False? The union of uncountably many measurable subsets of $[a,b]$ is measurable.

8. Prove: If E_1 and E_2 are measurable subsets of $[a,b]$, prove that the symmetric difference of E_1 and E_2 is measurable.

9. Prove: The set of rationals on $[a,b]$ is a measurable set and has measure zero and the set of irrationals on $[a,b]$ is a measurable set and has measure $b - a$.

2 Measurable Functions

Definition: Let $f : [a, b] \to \mathbb{R}$. The function f is a **measurable function** if for every real number s, the following set is measurable:

$$\{x \in [a, b] | f(x) > s\} = f^{-1}((s, \infty)).$$

For a function to be measurable, the inverse image of every open interval (s, ∞) must be measurable. Every continuous function $f : [a, b] \to \mathbb{R}$ has the property that the inverse image of every open set in $\mathbb{R}$ is open in $[a, b]$, i,e., if $G \subset \mathbb{R}$ is an open subset, then $f^{-1}(G)$ is an open subset of $[a, b]$.
Therefore, every continuous function $f : [a, b] \to \mathbb{R}$ is a measurable function.

The following theorem shows that sets of measure zero do not affect measurability for functions!

Theorem 2. *If $f : [a, b] \to \mathbb{R}$ is a measurable function and if $f(x) = g(x)$ almost everywhere ($f(x) = g(x)$ except for sets of measure zero on $[a, b]$), then g is also a measurable function.*

Proof. Let $s \in \mathbb{R}$ and let $E_1 = \{x \in [a, b] | g(x) > s\}$. (We want to show that E_1 is a measurable set.) Since f is a measurable function, the set $E_2 = \{x \in [a, b] | f(x) > s\}$ is a measurable set. Also, $f(x) = g(x)$ except for sets of measure zero on $[a, b]$, which means the symmetric difference of E_1 and E_2 (the set on which they are not equal) must have measure zero, $m(E_1 \Delta E_2) = 0$. Therefore, Theorem 1 shows that $m(E_2) = m(E_1)$, so E_1 is a measurable set. □

In-Class Problems

1. Prove: Dirichlet's discontinuous function defined on $[0, 1]$ is a measurable function.
2. Prove: If $f : [0, 1] \to \mathbb{R}$ is defined by $f(x) = \begin{cases} 1/x, & 0 < x < 1 \\ 5, & x = 0 \\ 7, & x = 1 \end{cases}$, then f is a measurable function.