Chapter 8 Sequences of Functions

In this chapter, we consider sequences of functions, $\{f_n(x)\}_{n=1}^{\infty}$, defined on a set $A, f_n : A \to \mathbb{R}$. We define pointwise convergence and uniform convergence of the functions to a function f(x) as $n \to \infty$. Whether the convergence is pointwise or uniform has important consequences for the limiting function f.

Examples:

- 1. $f_n(x) = \frac{x}{n}$ for $x \in \mathbb{R}$. Compute $\lim_{n \to \infty} f_n(x)$. 2. $g_n(x) = x^n$ for $x \in [-1, 1]$. Compute $\lim_{n \to \infty} g_n(x)$, if it exists.
- 3. $h_n(x) = \frac{x^2 + nx}{n}$ for $x \in \mathbb{R}$. Compute $\lim_{n \to \infty} h_n(x)$.
- 4. $F_n(x) = \frac{\sin(nx+n)}{n}$ for $x \in \mathbb{R}$. Compute $\lim_{n \to \infty} F_n(x)$.



Definition of Pointwise Convergence on A_0 : Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of functions with $f_n : A \to \mathbb{R}$. The sequence $\{f_n\}$ converges pointwise to a function f on $A_0 \subseteq A$, if for each $x \in A_0$, $\lim_{n\to\infty} f_n(x) = f(x)$. That is, for each $\epsilon > 0$ and each $x \in A_0$, there is a natural number $K(\epsilon, x)$ such that if $n \ge K(\epsilon, x)$, then

$$|f_n(x) - f(x)| < \epsilon$$

Definition of Uniform Convergence on A_0 Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of functions with $f_n : A \to \mathbb{R}$. The sequence $\{f_n\}$ converges uniformly to a function f on $A_0 \subseteq A$, if for each $\epsilon > 0$, there is a natural number $K(\epsilon)$ such that if $n \ge K(\epsilon)$, then

$$|f_n(x) - f(x)| < \epsilon$$
 for all $x \in A_0$.

Question: Which set(s) of functions $\{f_n\}, \{g_n\}, \{h_n\}, \{F_n\}$ converge uniformly on [0, 1]? on \mathbb{R} ?

We will define the **uniform norm** for a function $f : A \to \mathbb{R}$ and apply it to the sequence of functions $\{f_n\}$ defined on A. In general, a **norm** ρ defined on a set of real-valued functions \mathcal{F}_A is a mapping: $\rho : \mathcal{F}_A \to [0, \infty)$ with the following three properties:

- 1. $\rho(f+g) \le \rho(f) + \rho(g)$
- 2. $\rho(af) = |a|\rho(f)$ for any $a \in \mathbb{R}$.
- 3. If $\rho(f) = 0$, then f = 0 (zero function).

The uniform norm is applied to **bounded real-valued functions** $f : A \to \mathbb{R}$, where f(A) is bounded.

Definition: Let $f: A \to \mathbb{R}$ be a bounded function. The **uniform norm of** f **on** A is defined as

 $||f||_A := \sup\{|f(x)| : x \in A\}.$

It follows for any $\epsilon > 0$,

 $||f||_A \le \epsilon$ iff $|f(x)| \le \epsilon$ for all $x \in A$.

Show that the uniform norm satisfies the three norm properties. All of the previous examples f_n , g_n , h_n and F_n are bounded provided the domain is restricted to an interval of finite length, e.g., [-2, 2] or (0, 1).

- 1. The uniform norm for A = [-2, 2]: $||f_n||_A = \sup\{|x/n| : x \in [-2, 2]\} = \frac{2}{n}$.
- 2. The uniform norm for B = (0,1): $||g_n||_B = \sup\{|x^n| : x \in (0,1)\} = 1$.
- 3. The uniform norm for A = [-2, 2]: $||h_n||_A = \sup\{|x^2/n + x| : x \in [-2, 2]\} = \frac{4}{n} + 2.$
- 4. The uniform norm for A = [-2, 2]: $||F_n||_A = \sup\{|\sin(nx+n)/n| : x \in [-2, 2]\} = \frac{1}{n}$. In fact, we can extend the domain to \mathbb{R} and we get the same norm.

The following lemma allows us to use the uniform norm to check for uniform convergence.

Lemma 1. A sequence $\{f_n\}_{n=1}^{\infty}$ of bounded functions defined on A converges uniformly on A iff $||f_n - f||_A$ converges to zero.

Proof. Prove necessary and sufficient conditions.

Consider the previous four examples that have the following pointwise limit functions, f(x) = 0 g(x) = 0, h(x) = x and F(x) = 0. Which ones of the four sequences of functions converge uniformly to their pointwise limit?

1. $||f_n - f||_A = \sup\{|x/n - 0| : x \in [-2, 2]\} = \frac{2}{n} \to 0 \text{ as } n \to \infty.$

2.
$$||g_n - g||_B = \sup\{|x^n - 0| : x \in (0, 1)\} = 1.$$

3.
$$||h_n - h||_A = \sup\{|x^2/n + x - x| : x \in [-2, 2]\} = \frac{4}{n} \to 0 \text{ as } n \to \infty.$$

4. $||F_n - 0||_A = \sup\{|\sin(nx + n)/n| : x \in [-2, 2]\} = \frac{1}{n} \to 0 \text{ as } n \to \infty.$

Example: Prove: Let $\{f_n\}$ and $\{g_n\}$ be bounded uniformly convergent sequences. Then the sequence $\{f_ng_n\}$ is uniformly convergent (Problem # 23).

There is another criterion to show uniform convergence of a sequence of functions. This criterion is known as the Cauchy Criterion for Uniform Convergence. It is useful in cases where an explicit formula for the pointwise limit function is not known. This criterion will be used in the next section to verify results about the limiting function when the convergence is uniform! **Theorem 1** (Cauchy Criterion for Uniform Convergence). Let $\{f_n\}$ be a sequence of bounded functions defined on A. Then this sequence converges uniformly to a bounded function f iff for every $\epsilon > 0$, there is a natural number $H(\epsilon)$ such that for all $m, n \ge H(\epsilon)$, then

$$||f_m(x) - f_n(x)||_A < \epsilon$$
 for all $x \in A$.

Alternately,

$$|f_n(x) - f_m(x)| < \epsilon$$
 for all $x \in A$ and for $m, n \ge H(\epsilon)$

This criterion is used to prove the result about Interchange of Limit and Integral.

Example: (Triangle Functions) Consider the continuous triangle function $f_n(x)$, positive for $x \in (0, 2/n)$ and zero otherwise. The maximum height of the triangle function is at the midpoint x = 1/n with height $f_n(1/n) = n$. Thus, the area of the triangle function is (1/2)bh = (1/2)(2/n)(n) = 1. See Figure 8.2.1. Therefore, the definite integral of the triangle function over [0, 1], $\int_0^1 f_n(x) dx = 1$ for all n.

Consider the sequence of the triangle functions $\{f_n\}_{n=1}^{\infty}$. The sequence converges pointwise to the zero function f(x) = 0.

$$\int_0^1 f_n(x) \, dx = 1 \quad \text{and} \quad \int_0^1 f(x) \, dx = 0$$

The integral of the triangle functions does not converge to the integral of their pointwise limit! That is, for this example, **the limit and the integral cannot be interchanged**:

$$\lim_{n \to \infty} \int_0^1 f_n \neq \int_0^1 \lim_{n \to \infty} f_n.$$

Interchange of Limits

In this section, it is shown that if the sequence of functions converges uniformly to f, then the limit can be taken inside the integral (limit and integral can be interchanged). It is also shown that the uniform convergence of a sequence of continuous functions guarantees that the function to which they converge is also continuous.

Theorem 2 (Uniform Convergence of Continuous Functions). Let $\{f_n\}$ be a sequence of continuous functions defined on a set A that converges uniformly to a function f on A. Then f is continuous on A.

To show continuity of f on A, select an arbitrary point $c \in A$ and show if $\epsilon > 0$, there exists $\delta > 0$ such that for $|x - c| < \delta$, then $|f(x) - f(c)| < \epsilon$.

Proof. Let $\epsilon > 0$. Since the sequence of functions converges uniformly on A, there exists a natural number $K(\epsilon/3)$ such that

$$|f_n(x) - f(x)| < \frac{\epsilon}{3}$$

for all $x \in A$ and $n \ge K$. Let n be a fixed integer, $n \ge K$ and let c be an arbitrary point in A. By the triangle inequality,

$$\begin{aligned} |f(x) - f(c)| &= |f(x) - f_n(x) + f_n(x) - f_n(c) + f_n(c) - f(c)| \\ &\leq |f(x) - f_n(x)| + |f_n(x) - f_n(c)| + |f_n(c) - f(c)| \\ &< \epsilon/3 + |f_n(x) - f_n(c)| + \epsilon/3. \end{aligned}$$

Since f_n is continuous at x = c, there exists $\delta > 0$ such that $|f_n(x) - f_n(c)| < \epsilon/3$. Therefore, for this same δ , if $|x - c| < \delta$, then applying the previous inequalities implies $|f(x) - f(c)| < \epsilon$. Since c is an arbitrary point in A, f is continuous on A.

Question: Consider the sequence of functions $\{f_n\}$, $\{g_n\}$, $\{h_n\}$ and $\{F_n\}$ defined on the set A = [-1, 1]. Which sequence does NOT converge uniformly?

Exercise 8.2 # 1: Let $f_n(x) = \frac{x^n}{1+x^n}$. Each function f_n is continuous and bounded on [0,2]. See figure below.

- (a) Compute the pointwise limit f of $\{f_n\}$ on [0, 2]. Is the function f continuous?
- (b) Does the sequence converge uniformly?



The converse of the Theorem Uniform Convergence of Continuous Functions is FALSE.

FALSE: If f is continuous and f is the pointwise limit of a sequence of continuous functions $\{f_n\}$, then the sequence converges uniformly. See Exercises 8.1 # 5, # 15 with $f_n(x) = \frac{\sin(nx)}{(1+nx)}$, $x \in [0,2]$. See figure below.



There are two results regarding sequences of Riemann integrable functions. One important question is whether the limit can be moved inside the integral (interchange of limit and integral). The first result requires that the sequence be uniformly convergent. The second result requires a uniformly bounded sequence and the pointwise limit must also be Riemann integrable.

Theorem 3 (Interchange of Limit and Integral). Let $\{f_n\}$ be a sequence of Riemann integrable functions on [a,b] ($f_n \in \mathcal{R}[a,b]$) and suppose the sequence $\{f_n\}$ converges uniformly to a function f on [a,b]. Then $f \in \mathcal{R}[a,b]$ and

$$\lim_{n \to \infty} \int_a^b f_n(x) \, dx = \int_a^b \lim_{n \to \infty} f_n(x) \, dx.$$

Example: (Triangle Functions) The sequence of triangle functions $\{f_n\}$ on [0, 2/n] with height n, graphed in Figure 8.2.1, do not converge uniformly.

The following Bounded Convergence Theorem does not require uniform convergence of $\{f_n\}$ but it does require that the pointwise limit be Riemann integrable and the functions f_n must be uniformly bounded. It is also known as the Lebesgue's Dominated Convergence Theorem. The proof requires measure theory.

Theorem 4 (Bounded Convergence Theorem). Let $\{f_n\}$ be a sequence of Riemann integrable functions on [a,b] $(f_n \in \mathcal{R}[a,b])$ that converges pointwise to a Riemann integrable function f on [a,b] $(f \in \mathcal{R}[a,b])$. In addition, suppose that the functions $\{f_n\}$ are uniformly bounded, that is, there exists a constant B such that $|f_n(x)| \leq B$ for all n and $x \in [a,b]$, then

$$\lim_{n \to \infty} \int_a^b f_n(x) \, dx = \int_a^b \lim_{n \to \infty} f_n(x) \, dx.$$

Example: (Triangle Functions) The sequence of triangle functions $\{f_n\}$ on [0, 2/n] with height n do NOT satisfy the Bounded Convergence Theorem. They are not uniformly bounded,

$$\sup\{|f_n(x)| : x \in [0,1]\} = n \to \infty.$$

The last theorem is about the derivatives of a sequence. We need conditions on convergence of the derivatives. Uniform convergence of a sequence of differentiable functions $\{f_n\}$ to f does NOT imply that the function f is differentiable. We require the sequence of derivatives $\{f'_n\}$ converge uniformly to g, then f' = g(See Theorem 8.2.3). We will verify a stronger result that assumes the derivatives converge uniformly and are continuous and apply the Fundamental Theorems of Calculus.

Symbolically,

$$f_n \to f$$
, $f'_n \Rightarrow g$, and f'_n continuous, then $f' = g$.

Theorem 5 (Uniform Convergence of Derivatives). Let $f_n : [a,b] \to \mathbb{R}$ and let $\{f_n\}$ be a sequence that converges pointwise to f on [a,b] ($f_n \to f$). Suppose each derivative f'_n is continuous on [a,b] and the sequence of derivatives $\{f'_n\}$ converges uniformly to g on [a,b] ($f_n \Rightarrow g$). Then $f(x) - f(a) = \int_a^x g(t) dt$ and f'(x) = g(x) for all $x \in [a,b]$.

Proof. Since f_n is differentiable and f'_n is continuous on [a, b], then we can apply the Fundamental Theorem of Calculus (Part I) to f_n :

$$f_n(x) - f_n(a) = \int_a^x f'_n(t) dt$$

Take the limit as $n \to \infty$ of both sides: $\lim_{n\to\infty} (f_n(x) - f_n(a)) = f(x) - f(a) = \lim_{n\to\infty} \int_a^x f'_n(t) dt$. Because the sequence $\{f'_n\}$ converges uniformly to g and each f'_n is Riemann integrable, by the Theorem on Interchange of Limit and Integral, the limit can be brought inside the integral:

$$f(x) - f(a) = \lim_{n \to \infty} \int_{a}^{x} f'_{n}(t) \, dt = \int_{a}^{x} \lim_{n \to \infty} f'_{n}(t) \, dt = \int_{a}^{x} g(t) \, dt$$

Also, g is continuous on [a, b]. (Why?) By the Fundamental Theorem of Calculus (Part II), f'(x) = g(x) for all $x \in [a, b]$.

Exercise 8.2 # 10: Let $g_n(x) = e^{-nx}/n$ for $x \ge 0$. What can you say about $\lim g_n$ and $\lim g'_n$?

Series of Functions

The theorems on sequences of functions apply to series of functions:

$$\sum_{n=1}^{\infty} f_n(x).$$

The expression for the infinite series is also written as $\sum f_n$ or as $\sum_{n=1}^{\infty} f_n$. Convergence of the series is defined in terms of convergence of its partial sums (Chapter 3). Pointwise convergence, uniform convergence and absolute convergence of a series of functions are defined below.

Definition: Let $f_n : A \to \mathbb{R}$. The sequence of partial sums of the infinite series $\sum f_n$ is the sequence $\{s_n\}$ where $s_n : A \to \mathbb{R}$ is defined by

$$s_{1}(x) = f_{1}(x)$$

$$s_{2}(x) = f_{1}(x) + f_{2}(x)$$

$$\vdots$$

$$s_{n}(x) = f_{1}(x) + \dots + f_{n}(x) = \sum_{k=1}^{n} f_{k}(x).$$

The infinite series $\sum f_n$ converges pointwise to f on A if the sequence of partial sums $\{s_n\}$ converge pointwise to f on A, i.e.,

$$\lim_{n \to \infty} s_n(x) = \lim_{n \to \infty} \sum_{k=1}^n f_k(x) = f(x) \text{ for } x \in A.$$

The infinite series $\sum f_n$ converges uniformly to f on A if the sequence of partial sums $\{s_n\}$ converges uniformly to f on A, i.e., for every $\epsilon > 0$, there exists a natural number $K(\epsilon)$ such that for $n \ge K(\epsilon)$,

$$|s_n(x) - f(x)| = \left|\sum_{k=1}^n f_k(x) - f(x)\right| < \epsilon \text{ for all } x \in A$$

The infinite series $\sum f_n$ converges absolutely to f on A if the infinite series $\sum |f_n|$ converges pointwise to a function on A. But $\sum f_n(x) = f(x)$ does NOT imply $\sum |f_n(x)| = |f(x)|$, e.g., $\sum (-0.5)^n$

Example: Recall the formula for a geometric series

$$\sum_{n=0}^{\infty} ar^n = a(1+r+r^2+\dots+r^n+\dots) = \begin{cases} \frac{a}{1-r} & |r|<1, \\ \text{diverges}, & |r| \ge 1. \end{cases}$$

Now consider the continuous functions $f_n(x) = x^n$ and the infinite series

$$\sum_{n=1}^{\infty} f_n(x) = \sum_{n=1}^{\infty} x^n = x + x^2 + \dots + x^n + \dots$$

For |x| < 1, that is, for x in the open interval (-1, 1) this series converges pointwise to a function f(x).

- (a) What is the function f(x)?
- (b) Does this series converge absolutely for |x| < 1?

We will discuss uniform convergence for this series after we discuss the Weierstrass M-Test.

The previous theorems on Cauchy Criterion for Uniform Convergence, Uniform Convergence of Continuous Functions, Interchange of Limit and Integral, and Uniform Convergence of Derivatives (Theorems 1, 2, 3 and 5) apply to series of functions. To verify an infinite series converges, it is necessary to show that its sequence of partial sums $\{s_n\}$ converges. The Cauchy Criterion for a sequence of functions $\{f_n\}$: for $m, n > H(\epsilon)$, $|f_m(x) - f_n(x)| < \epsilon$. For a sequence of partial sums $\{s_n\}$, the criterion is $|s_m(x) - s_n(x)| < \epsilon$ which for m > n is written as follows:

$$\left|\sum_{k=1}^{m} f_k(x) - \sum_{k=1}^{n} f_k(x)\right| = |f_{n+1}(x) + f_{n+2}(x) + \dots + f_m(x)| < \epsilon.$$

The four Theorems 1, 2, 3, and 5 are stated in terms of the convergence of the infinite series $\sum f_n$.

Theorem 6 (Cauchy Criterion for Uniform Convergence (Series)). Let $f_n : A \to \mathbb{R}$. The infinite series $\sum f_n$ converges uniformly to a function f iff for every $\epsilon > 0$, there is a natural number $H(\epsilon)$ such that for all $m > n \ge H(\epsilon)$, then

$$|f_{n+1}(x) + f_{n+2}(x) + \dots + f_m(x)| < \epsilon \text{ for all } x \in A.$$

Theorem 7 (Uniform Convergence of Continuous Functions (Series)). Let $f_n : A \to \mathbb{R}$ be continuous for each $n \in \mathbb{N}$. If the infinite series $\sum f_n$ converges uniformly to a function f on A, then f is continuous on A.

Theorem 8 (Interchange of Limit and Integral (Series)). Let $f_n : [a,b] \to \mathbb{R}$ be Riemann integrable functions on [a,b] ($f_n \in \mathcal{R}[a,b]$) and suppose the infinite series $\sum f_n$ converges uniformly to a function f on [a,b]. Then $f \in \mathcal{R}[a,b]$ and

$$\lim_{n \to \infty} \int_{a}^{b} \sum_{k=1}^{n} f_{k}(x) \, dx = \int_{a}^{b} \lim_{n \to \infty} \sum_{k=1}^{n} f_{k}(x) \, dx = \int_{a}^{b} f(x) \, dx.$$

Theorem 9 (Uniform Convergence of Derivatives (Series)). Let $f_n : [a, b] \to \mathbb{R}$ and let $\sum f_n$ converge pointwise to f on [a, b]. Suppose each derivative f'_n is continuous on [a, b] and the series of derivatives $\sum f'_n$ converges uniformly to g on [a, b]. Then $f(x) - f(a) = \int_a^x g(t) dt$ and

$$\sum_{n=1}^{\infty} f'_n(x) = f'(x) = g(x), \text{ for all } x \in [a, b].$$

The following theorem is called the Weierstrass M-Test. This test gives sufficient conditions for uniform convergence of a series of functions.

Theorem 10 (Weierstrass M-Test). Let $f_n : A \to \mathbb{R}$ and let $\{M_n\}$ be a sequence of positive real numbers such that the absolute value of f_n satisfies $|f_n(x)| \leq M_n$ for all $x \in A$ and $n \in \mathbb{N}$. If the infinite series $\sum M_n$ is convergent, then the infinite series $\sum f_n$ converges uniformly for all $x \in A$.

To verify this theorem, two Cauchy convergence criteria are applied to series, Theorem 3.7.4 to $\sum M_n$ and Theorem 6 given above or Theorem 9.4.5 to $\sum f_n(x)$. Cauchy Criteria are iff statements.

Proof. Let $\epsilon > 0$. The convergence of the positive series $\sum M_n$ implies the Cauchy Criterion holds. There exists a natural number $K(\epsilon)$ such that for $m > n \ge K(\epsilon)$, then

$$M_{n+1} + M_{n+2} + \dots + M_m < \epsilon.$$

Consider $\sum f_n(x)$ and show the sequence of partial sums $\{s_n(x)\}$ satisfy the Cauchy Criterion for Uniform Convergence. For $m > n > K(\epsilon)$,

$$\begin{aligned} |s_m(x) - s_n(x)| &= |f_{n+1}(x) + f_{n+2}(x) + \dots + f_m(x)| \\ &\leq |f_{n+1}(x)| + |f_{n+2}(x)| + \dots + |f_m(x)| \\ &< M_{n+1} + M_{n+2} + \dots + M_m < \epsilon \end{aligned}$$

for all $x \in A$. The preceding inequality shows that the Cauchy Criterion holds for Uniform Convergence of $\sum f_n(x)$ on A.

Example: Now we return to the infinite series $\sum x^n$. This series converges pointwise and absolutely to f(x) = x/(1-x) on the open interval (-1, 1). Consider a closed and bounded interval $[-a, a] \subseteq (-1, 1)$. Then $|f_n(x)| = |x^n| \le a^n$ for all $x \in [-a, a]$. For $M_n = a^n$, it is a geometric series $\sum a^n$ which converges for |a| < 1. By the Weierstrass M-Test, the infinite series $\sum x^n$ converges uniformly for $x \in [-a, a] \subseteq (-1, 1)$.

(a) The Weierstrass M-Test does not apply on the open interval (-1, 1). Why?

(b) Show that the infinite series $\sum x^n$ does not converge uniformly on (-1, 1). (*Hint:* Consider $||s_m - s_n||_{(-1,1)}$ for m = 2n.)

Exercise: Note that the Cauchy Criterion for Uniform Convergence of Series does not mention bounded functions. Prove that if the functions f_n are bounded, $|f_n(x)| \leq M_n$ for all $x \in A$, and the series $\sum f_n$ converges uniformly on A to a function f, then f is bounded. (*Hint:* Write $|f(x)| = |f(x) - s_n(x) + s_n(x)|$.)

Tests for Convergence and Divergence of Series

Please refer to the summary on Tests for Convergence and Divergence of Series $\sum a_n$ that you learned in Calculus II. We will review definitions of conditional and absolute convergence, alternating series, ratio and root tests, comparison tests, and specific series including geometric series, harmonic series and *p*-series. You may use the summary on Tests for Convergence and Divergence of Series for Exam # 2.

These tests for convergence and divergence are discussed in Sections 9.1 through 9.3. Some new definitions will be introduced such as "grouping of series" and "rearrangement of series", as well as some new theorems about convergence of series, such as Dirichlet's Test and Abel's Test. Then we will return to section 9.4 and discuss power series, Taylor series, and Fourier series.