Math 4606. Fall 2006.

Solutions to Homework 8

Note: this homework assignment is not collected.

Section 2.1.

Problem 1. Suppose f is differentiable on an interval I and that $f'(x) > 0$ for all $x \in I$ except for finitely many points at which $f'(x) = 0$. Show that f is strictly increasing on I.

Solution. Note that by using the mean value theorem (same way as in Theorem 2.8 of the textbook), we have:

$$(1) \quad \textit{If } f \textit{ is continuous on } [a,b] \textit{ and } f'(x) > 0 \textit{ for all } x \in (a,b), \\ \textit{then } f \textit{ is strictly increasing on } [a,b].$$

Let $\underline{a, b \in I \text{ and } a < b}$. Denote $C = \{x \in I : f'(x) = 0\}$, then $f'(x) > 0$ for all $x \in I \setminus C$.

Case 1: $(a,b) \cap C = \emptyset$. Then $f'(x) > 0$ for all $x \in (a,b)$ and by (1) we have $\underline{f(a) < f(b)}$.

Case 2: $(a,b) \cap C \neq \emptyset$. Then there are $N \geq 1$ and $c_i \in C$, for $i \leq N$, such that $a < c_1 < c_2 < \ldots < c_N < b$. Since f is continuous at a, b and c_i, for $i \leq N$, we apply (1) to $[a, c_1]$, $[c_N, b]$ and each $[c_i, c_{i+1}]$ for $i \leq N-1$, and obtain $f(a) < f(c_1) \leq f(c_2) \leq \ldots \leq f(c_N) < f(b)$. (Note that if $N = 1$ we only consider $[a, c_1]$ and $[c_1, b]$). Therefore $\underline{f(a) < f(b)}$. The proof is complete.

Problem 2. Let

$$f(x) = \begin{cases} x^2 \sin(\frac{1}{x}) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Show that $f'(x)$ exists for all $x \in \mathbb{R}$, but $f'(x)$ is discontinuous at $x = 0$.

Solution. Easy to see that for $x \neq 0$,

$$(2) \qquad f'(x) = 2x\sin(\frac{1}{x}) + x^2(-\frac{1}{x^2})\cos(\frac{1}{x}) = 2x\sin(\frac{1}{x}) - \cos(\frac{1}{x}).$$

At $x = 0$, consider $h \neq 0$ and

$$(3) \qquad \left|\frac{f(h) - f(0)}{h - 0}\right| = |h \sin(\frac{1}{h})| \leq |h| \to 0 \text{ as } h \to 0.$$

Hence $\lim_{h\to 0} \frac{f(h)-f(0)}{h-0} = 0$ which gives $f'(0) = 0$. Thus f is differentiable every where.

Considering (2), we have $\lim_{x\to 0} 2x\sin(\frac{1}{x}) = 0$ and $\lim_{x\to 0}\cos(\frac{1}{x})$ does not exist (why?). Hence $\lim_{x\to 0} f'(x)$ does not exists (why?). Therefore $f'(x)$ is discontinuous at 0.

Problem 5. Let f be continuous on $[a, n]$ and differentiable on (a, b). Suppose that the right-hand limit

$$\lim_{x\to a+} f'(x) = L \text{ exists.} \tag{4}$$

Show that the right-hand derivative $f'_+(a)$ defined by

$$f'_+(a) = \lim_{h\to 0+} \frac{f(a+h) - f(a)}{h} \tag{5}$$

also exists and equals L.

Solution. For $h \in (0, b-a)$, using the mean value theorem we have

$$\frac{f(a+h) - f(a)}{h} = f'(c_h)$$

for some $c_h \in (a, a+h)$. When $h \to 0+$, we have $c_h \to a+$ and it follows from (4) that $\lim_{h\to 0+} f'(c_h) = L$. Therefore

$$\lim_{h\to 0+} \frac{f(a+h) - f(a)}{h} = \lim_{h\to 0+} f'(c_h) = L$$

Thus $f'_+(a) = L$.

Problem 6a. Let f be three times differentiable on an open interval I containing a. Show that

$$\lim_{h\to 0} \frac{f(a+2h) - 2f(a+h) + f(a)}{h^2} = f''(a). \tag{6}$$

Solution. There is $\delta > 0$ such that $(a - 3\delta, a + 3\delta) \subset I$. In the following, we only consider $|h| < \delta$. For $x \in (a - 2\delta, a + 2\delta)$, define

$$g(x) = \frac{f(x+h) - f(x)}{h}.$$

We have $g'(x) = \frac{f'(x+h) - f'(x)}{h}$, for $x \in (a - 2\delta, a + 2\delta)$.

First consider $h > 0$. Note that $a + 2h, a + h, a \in (a, a + 2\delta)$, then

$$\frac{f(a+2h) - 2f(a+h) + f(a)}{h^2} = \frac{g(a+h) - g(a)}{h}.$$

Applying the mean value theorem to g,

$$\frac{g(a+h) - g(a)}{h} = g'(y_h) = \frac{f'(y_h + h) - f'(y_h)}{h},$$

where $y_h \in (a, a+h) \subset (a, a+\delta)$ which gives $y_h + h \in (a, a+2h) \subset (a, a+2\delta)$. Applying the mean value theorem again to f', we obtain

$$\frac{f(a+2h) - 2f(a+h) + f(a)}{h^2} = \frac{f'(y_h + h) - f'(y_h)}{h} = f''(z_h),$$

where $z_h \in (y_h, y_h + h) \subset (a, a+2h) \subset I$, hence $z_h \to a+$ as $h \to 0+$. Since f is three times differentiable, we have f'' is differentiable hence continuous, thus

$$\lim_{h \to 0+} \frac{f(a+2h) - 2f(a+h) + f(a)}{h^2} = \lim_{h \to 0+} f''(z_h) = f''(a).$$

Similarly, one can prove the other one-sided limit

$$\lim_{h \to 0-} \frac{f(a+2h) - 2f(a+h) + f(a)}{h^2} = f''(a).$$

Therefore (6) follows.