# Math 4606. Fall 2006.

## Solutions to Homework 7

**Section 1.7.**

*Problem 4.* Suppose $S_1$ and $S_2$ are connected subsets in $\mathbb{R}^n$ with $S_1 \cap S_2 \neq \emptyset$.

(a) Show that $S_1 \cup S_2$ is notnnected.

(b) Is $S_1 \cap S_2$ connected?

*Solution.* (a) Suppose $S = S_1 \cup S_2$ is not connected. Then there is a disconnection $(V_1, V_2)$ of $S$, ie.,

$$V_1, V_2 \neq \emptyset, \quad S = V_1 \cup V_2, \quad V_1 \cap \bar{V}_2 = V_2 \cap \bar{V}_1 = \emptyset. \tag{1}$$

Since $S_1 \cap S_2 \neq \emptyset$, there is $a \in S_1 \cap S_2$. Also $a \in S = V_1 \cup V_2$, we have $a \in V_1$ or $a \in V_2$. Without loss of generality, we assume $a \in V_1$. Since

$$\emptyset \neq V_2 = V_2 \cap S = V_2 \cap (S_1 \cup S_2) = (V_2 \cap S_1) \cup (V_2 \cap S_2),$$

we have $V_2 \cap S_1 \neq \emptyset$ or $V_2 \cap S_2 \neq \emptyset$.

*Case 1:* $V_2 \cap S_1 \neq \emptyset$. Let $T_1 = V_1 \cap S_1$ and $T_2 = V_2 \cap S_1$. We have $T_2 \neq \emptyset$ by assumption, and $T_1 \neq \emptyset$ since it contains $a$. Also,

$$T_1 \cup T_2 = T_1 = (V_1 \cap S_1) \cup (V_2 \cap S_1) = (V_1 \cup V_2) \cap S_1 = S \cap S_1 = S_1.$$

Furthermore, $T_1 \subset V_1, T_2 \subset V_2$, hence $T_1 \cap \bar{T}_2 \subset V_1 \cap \bar{V}_2 = \emptyset$ which implies $T_1 \cap \bar{T}_2 = \emptyset$. Similarly, $T_2 \cap \bar{T}_1 = \emptyset$. Therefore $(T_1, T_2)$ is a disconnection of $S_1$ which contradicts the fact that $S_1$ is connected.

*Case 2:* $V_2 \cap S_2 \neq \emptyset$. Let $T_1 = V_1 \cap S_2$ and $T_2 = V_2 \cap S_2$. Using these sets, we can prove similarly that $S_2$ is disconnected which is absurd.

We find contradiction in both cases, hence we conclude that $S$ is connected.

(b) Break a (very thin) doughnut into halves and slide one slightly over the other. This gives a picture of two sets in $\mathbb{R}^2$ which are connected with non-empty intersection but their intersection is disconnected. Indeed, that intersection is the union of two regions staying far apart from each other. This is not a real proof but we can easily write down many other examples. For instance, in $\mathbb{R}^2$, take $S_1$ to be the unit circle and $S_2$ to be the $x$-axis.

*Problem 10.* Let $S$ be a connected set in $\mathbb{R}^2$ that contains $(1,3)$ and $(4,-1)$. Show that $S$ contains at least one point on the line $y = x$.

*Solution.* Let $f(x,y) = y - x$. Then $f$ is continuous. We have $f(1,3) = 2$ and $f(4,-1) = -5$. Note that $f(4,-1) < 0 < f(1,3)$. Since $S$ is connected containing $(1,3)$ and $(4,-1)$, and $f$ is continuous, then by the intermediate value theorem there is $(a,b) \in S$ such that $f(a,b) = 0$, hence $a = b$ or the point $(a,b)$ of $S$ belongs to the line $y = x$.

Note: A similar example was given in lectures.

**Section 1.8.**

*Problem 3.* Let $S \subset \mathbb{R}^n$ and $f, g : S \to \mathbb{R}^m$ be uniformly continuous. Show that $f + g$ is uniformly continuos on $S$.

*Solution.* Let $\varepsilon > 0$. Since $f$ is uniformly continuous, there is $\delta_1 > 0$, so that for $x, y \in S$ with $|x - y| < \delta_1$ we have $|f(x) - f(y)| < \varepsilon/2$. Similarly, there is $\delta_2 > 0$, so that for $x, y \in S$ with $|x - y| < \delta_2$ we have $|g(x) - g(y)| < \varepsilon/2$. Let $\delta = \min\{\delta_1, \delta_2\} > 0$. For any $x, y \in S$ such that $|x - y| < \delta$, we have $|x - y| < \delta_1$, $|x - y| < \delta_2$ and by the triangle inequality

$$\begin{aligned} |(f+g)(x) - (f+g)(y)| &= |f(x) - f(y) + g(x) - g(y)| \\ &\leq |f(x) - f(y)| + |g(x) - g(y)| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2}. \end{aligned}$$

Hence $|(f+g)(x) - (f+g)(y)| < \varepsilon$. Therefore, $f + g$ is uniformly continuos on $S$.

*Problem 4.* (a) Suppose $S \subset \mathbb{R}^n$ and $f : S \to \mathbb{R}^m$ is uniformly continuous and $\{x_k\}$ is a Cauchy sequence in $S$. Show that $\{f(x_k)\}$ is a Cauchy sequence in $\mathbb{R}^m$.

(b) Give an example of a Cauchy sequence in $(0, \infty)$ and a continuous function $f : (0, \infty) \to \mathbb{R}$ such that $\{f(x_k)\}$ is not Cauchy.

*Solution.* (a) Let $\varepsilon > 0$. Since $f$ is uniformly continuous, there is $\delta > 0$, so that for $x, y \in S$:

$$|x - y| < \delta \implies |f(x) - f(y)| < \varepsilon. \tag{2}$$

Since $\{x_k\}$ is Cauchy, there is $K \in \mathbb{N}$ such that if $k, j > K$ we have $|x_k - x_j| < \delta$. Now with such $K \in \mathbb{N}$ and for any $k, j > K$, we again have $|x_k - x_j| < \delta$ and hence (2) implies $|f(x_k) - f(x_j)| < \varepsilon$. Therefore, $\{f(x_k)\}$ is a Cauchy sequence in $\mathbb{R}^m$.

(b) We can take $f(x) = 1/x$ and $x_k = 1/k$.