Math 4606. Fall 2006.
Solutions to Homework 3

Section 1.3.

Problem 1c. Let

$$f(x,y) = \frac{x^4y^4}{(x^2+y^4)^3}, \text{ for } (x,y) \neq (0,0).$$

Show that $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

Solution. This problem was solved in class. Briefly,

$$f(0,y) = 0, \text{ for all } y \neq 0.$$

For $c \in \mathbb{R}, y \neq 0$,

$$f(cy^2, y) = \frac{c^4y^{12}}{y^{12}(c^2+1)^3} = \frac{c^4}{(c^2+1)^3}.$$

Take, for example, $c = 1$ we have

$$\lim_{y\to 0} f(0,y) \neq \lim_{y\to 0} f(y^2, y),$$

while $\lim_{y\to 0}(0,y) = (0,0) = \lim_{y\to 0}(y^2,y)$.

Problem 2. Show that $\lim_{(x,y)\to(0,0)} f(x,y) = 0$.

a.

$$f(x,y) = \frac{x^2y^2}{x^2+y^2}.$$

Solution. For $(x,y) \neq (0,0)$, we have

$$0 \leq f(x,y) = x^2\frac{y^2}{x^2+y^2} \leq x^2\frac{y^2+x^2}{x^2+y^2} = x^2.$$

Since $\lim_{(x,y)\to(0,0)} x^2 = 0$, then by the squeezing property we obtain $\lim_{(x,y)\to(0,0)} f(x,y) = 0$.

b.

$$f(x,y) = \frac{3x^5 - xy^4}{x^4+y^4}.$$

Solution. For $(x,y) \neq (0,0)$, we use the triangle inequality:

$$0 \leq |f(x,y)| = \frac{|3x^5 - xy^4|}{x^4+y^4} \leq \frac{3|x|x^4}{x^4+y^4} + \frac{|x|y^4}{x^4+y^4} \leq 3|x| + |x| = 4|x|.$$

Since $\lim_{(x,y)\to(0,0)} 4|x| = 0$, then by the squeezing property we obtain $\lim_{(x,y)\to(0,0)} |f(x,y)| = 0$. Therefore $\lim_{(x,y)\to(0,0)} f(x,y) = 0$.

Problem 6. Let

$$f(x) = \begin{cases} x & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

a. Show that f is continuous at 0.

Solution. Note that $f(0) = 0$. Let $\varepsilon > 0$ be arbitrary. Take $\delta = \varepsilon$. Let $x \in \mathbb{R}$ such that $|x| < \delta$.

If x is rational then $|f(x) - f(0)| = |x - 0| = |x| < \delta = \varepsilon$.

If x is irrational then $|f(x) - f(0)| = 0 < \varepsilon$.

In both cases, we have $|f(x) - f(0)| < \varepsilon$ whenever $|x| < \delta$. Therefore, f is continuous at 0.

b. Show that f is discontinuous at any point $a \neq 0$.

Solution. Let $a \neq 0$.

Case 1: a is rational, then $f(a) = a$. Take $\varepsilon_0 = |a| > 0$. Let $\delta > 0$ be arbitrary. Choose x_δ to be an irrational number in the interval $(a - \delta, a + \delta)$, then we have $|x_\delta - a| < \delta$ and

$$|f(x_\delta) - f(a)| = |0 - a| = |a| \geq \varepsilon_0.$$

Therefore f is not continuous at a.

Case 2: a is irrational, then $f(a) = 0$. Take $\varepsilon_0 = |a|/2 > 0$. Let $\delta > 0$ be arbitrary. Choose x_δ to be a rational number in the interval $(a - \delta, a + \delta) \cap (a - \varepsilon_0, a + \varepsilon_0)$, then we have $|x_\delta - a| < \delta$. Also, $|x_\delta - a| < \varepsilon_0$, we obtain

$$|f(x_\delta) - f(a)| = |x_\delta| \geq |a| - |x_\delta - a| \geq |a| - \varepsilon_0 = |a|/2 = \varepsilon_0.$$

Therefore f is not continuous at a.

Problem 7. Let $f(x) = 1/q$ if $x = p/q$, where p, q are integers having no common factors, $q > 0$, and $f(x) = 0$ if x is irrational.

a. Show that f is discontinuous at rational numbers.

Solution. Let $a = p/q$ be a rational number where p, q are as above. Let $\varepsilon_0 = 1/q > 0$. Let $\delta > 0$ be arbitrary. Choose x_δ to be an irrational number in the interval $(a - \delta, a + \delta)$ then we have $|x_\delta - a| < \delta$ and

$$|f(x_\delta) - f(a)| = |0 - \frac{1}{q}| = \frac{1}{q} = \varepsilon_0.$$

Therefore f is discontinuous at a.

b. Show that f is continuous at irrational numbers.

Solution. Let a be an irrational number, then $f(a) = 0$. We need to show that f is continuous at a. Note that for any irrational number x, we have $|f(x) - f(a)| = 0$ which is less than any positive number ε. Therefore it suffices to show that

$$\forall \varepsilon > 0, \exists \delta > 0, \forall x = p/q \in \mathbb{Q}, |x - a| < \delta \implies |1/q| < \varepsilon, \tag{1}$$

where p, q are integers having no common factors and $q > 0$. We will prove (1) by contradiction method. Suppose (1) is not true, then the negation of (1) holds:

$$\exists \varepsilon_0 > 0, \forall \delta > 0, \exists x_\delta = p_\delta/q_\delta \in \mathbb{Q}, |x_\delta - a| < \delta \text{ and } |1/q_\delta| \geq \varepsilon_0. \tag{2}$$

For $k \in \mathbb{N}$, take $\delta = 1/k$ in (2), then there is $x_k = p_k/q_k \in \mathbb{Q}$ such that

$$|x_k - a| < 1/k \text{ and } |q_k| \leq 1/\varepsilon_0. \tag{3}$$

By (3) and triangle inequality,

$$|p_k|/|q_k| = |x_k| \leq |a| + |x_k - a| \leq |a| + (1/k) \leq |a| + 1.$$

This and (3) yield $|p_k| \leq (|a| + 1)|q_k| \leq (|a| + 1)/\varepsilon_0$. Therefore, the integers p_k and q_k are bounded, hence each set

$$A = \{p_k, k \in \mathbb{N}\}, \quad B = \{q_k, k \in \mathbb{N}\}$$

has only finitely many elements. Consequently, the set $S = \{p/q, p \in A, q \in B\}$ has only finitely many elements. Note that $x_k \in S$ for all k and $S \subset \mathbb{Q}$. Since a is irrational, $a \notin S$. For each $x \in S$, $|a - x| > 0$. Since S is finite, then $d = \min\{|a - x| : x \in S\} > 0$. Therefore, for any k, $|x_k - a| \geq d$, this contradicts the fact that $|x_k - a| \leq 1/k \to 0$ as $k \to \infty$. Thus, we must have (1). The proof is complete.