Math 4606. Fall 2006.
Solutions to Homework 2

Section 1.1.

Problem 1. Given $x, y \in \mathbb{R}^n$.

a. Show that $|x+y|^2 = |x|^2 + 2x \cdot y + |y|^2$.

Solution. Using the definition of the norm in $\mathbb{R}^n$ and the commutativity and associativity of the dot product, we have

$$\begin{aligned} |x+y|^2 &= (x+y)\cdot(x+y) = x\cdot x + x\cdot y + y\cdot x + y\cdot y \\ &= |x|^2 + x\cdot y + x\cdot y + |y|^2 \\ &= |x|^2 + 2x\cdot y + |y|^2. \end{aligned}$$

b. Show that $|x+y|^2 + |x-y|^2 = 2(|x|^2 + |y|^2)$.

Solution. Using part (a),

$$|x+y|^2 = |x|^2 + 2x\cdot y + |y|^2,$$

$$|x-y|^2 = |x+(-y)|^2 = |x|^2 + 2x\cdot(-y) + |-y|^2 = |x|^2 - 2x\cdot y + |y|^2.$$

Then sum the above two identities,

$$|x+y|^2 + |x-y|^2 = |x|^2 + 2x\cdot y + |y|^2 + |x|^2 - 2x\cdot y + |y|^2 = 2|x|^2 + 2|y|^2.$$

Problem 6. Show that $\big||a| - |b|\big| \le |a-b|$ for every $a, b \in \mathbb{R}^n$.

Solution. Let a and b be any vectors in $\mathbb{R}^n$. By the triangle inequality,

$$|a| = |a-b+b| \le |a-b| + |b|,$$

hence $|a| - |b| \le |a-b|$. Similarly

$$|b| - |a| = |b-a+a| - |a| \le |b-a| + |a| - |a| = |a-b|.$$

Since both $|a|-|b|$ and its opposite $-(|a|-|b|) = |b|-|a|$ are less than or equaled to $|a-b|$, so is its absolute value $\big||a|-|b|\big|$.

Section 1.2.

Problem 2. Let S an a subset of $\mathbb{R}^n$. a. Show that S^{int} is open.

Solution. Let $x \in S^{\text{int}}$ be given. Since x is an interior point of S, then by definition there is $r > 0$ such that the ball $B(r, x)$ is a subset of S. We will show that this ball, indeed, is a subset of S^{int}.

Let $y \in B(r,x)$. Since $B(r,x)$ is an open set (Example 1 in the text which was also done in class) there is $r_1 > 0$ such that $B(r_1, y) \subset B(r,x)$ which implies that $B(r_1,y) \subset S$, hence y is an interior point of S, or $y \in S^{\text{int}}$. Thus $B(r,x) \subset S^{\text{int}}$.

We have just showed that every point $x \in S^{\text{int}}$ is an interior point of S^{int} (let not be confused with interior point of S). Then by Proposition 1.4 (in the text), S^{int} must be open. The proof is complete.

b. Show that ∂S is closed.

Solution. We know that $\partial S = \mathbb{R}^n \setminus \{S^{\text{int}} \cup (S^c)^{\text{int}}\}$. By part (a), S^{int} and $(S^c)^{\text{int}}$ are open, so is there union $S^{\text{int}} \cup (S^c)^{\text{int}}$. Thefore ∂S is closed, by Proposition 1.4(b).

c. Show that $\bar{S}$ is closed.

Solution. We know that $\bar{S} = \mathbb{R}^n \setminus (S^c)^{\text{int}}$. By part (a), $(S^c)^{\text{int}}$ is open, therefore $\bar{S}$ is closed.

Problem 4. Let S_1 and S_2 be two closed subsets of $\mathbb{R}^n$. Show that $S_1 \cup S_2$ and $S_1 \cap S_2$ are closed.

Solution. By Proposition 1.4(b), S_1^c and S_2^c are open, hence their union $S_1^c \cup S_2^c$ and and intersection $S_1^c \cap S_2^c$ are open. Since $(S_1 \cup S_2)^c = S_1^c \cap S_2^c$ and $(S_1 \cap S_2)^c = S_1^c \cup S_2^c$, we have $(S_1 \cup S_2)^c$ and $(S_1 \cap S_2)^c$, therefore $S_1 \cup S_2$ and $S_1 \cap S_2$ are closed.

Problem 6. Give an example of closed sets S_k, $k \in \mathbb{N}$, such that $\cup_{k=1}^{\infty} S_k$ is not closed.

Solution. There are many examples. Here is one of them. Let $S_k = [0, 2 - \frac{1}{k}]$, $k \in \mathbb{N}$, be closed set in $\mathbb{R}$. Their union $S = \cup_{k=1}^{\infty} S_k$ is $[0,2)$ is not closed. How to prove $S = [0,2)$? First $S_k \subset [0,2)$ for all k, so $S \subset [0,2)$. Suppose $[0,2)$ is not a subset of S then there is $x_0 \in [0,2)$ but $x_0 \notin S$. This implies $x_0 \notin S_k$ for all k. Take a large natural number $K > 1/(2 - x_0)$, then $x_0 < 2 - (1/K)$ hence $x_0 \in S_K$, contradiction. Therefore $[0,2) \subset S$. Thus $S = [0,2)$.

Question: Why isn't $[0,2)$ closed?