Chapter 6

Infinite series

We briefly review this chapter in order to study series of functions in chapter 7. We cover from the beginning to Theorem 6.17 in the text excluding Theorem 6.6 and Rabbe's test (Theorem 6.16) from section 6.2.

Chapter 7

Functions Defined by Series and Integrals

We focus on sequences and series of functions and their uniform convergence. We only present a summary for Section 7.3. The first two sections are rearrangements of those in the text.

7.1 Sequences of functions

Let S be a subset of $\mathbb{R}^n$ and for each $k \in \mathbb{N}$, f_k is a function from S to $\mathbb{R}^m$. Let $f : S \to \mathbb{R}^m$. We want to understand what "f_k converges to f as $k \to \infty$" means and the relations between f_k and f.

Pointwise convergence. We say $f_k \to f$ *pointwise* on S if

$$\forall x \in S, \lim_{k\to\infty} f_k(x) = f(x).$$

However, this kind of convergence is not good enough when we consider the continuity, integrability or differentiability of f.

Uniform convergence. We say $\{f_k\}$ *converges uniformly on* S to f if

$$\forall \varepsilon > 0, \exists K > 0, \forall k \in \mathbb{N} : \big[k > K \implies (\forall x \in S, |f_k(x) - f(x)| < \varepsilon)\big].$$

Some books use the notation: $f_k \rightrightarrows f$.

We say $\{f_k\}$ is *uniformly convergent* if its converges uniformly to some function f.

Note that if $f_k \to f$ uniformly then obviously $f_k \to f$ pointwise.

Theorem 7.1. *The sequence $\{f_k\}$ converges to f uniformly on S if and only if*

$$\lim_{k\to\infty} M_k = 0, \tag{7.1}$$

where

$$M_k = \sup\{|f_k(x) - f(x)| : x \in S\}. \tag{7.2}$$

Corollary 7.2. *If there are $C_k \geq 0$ such that*

$$|f_k(x) - f(x)| \leq C_k, \text{ for all } x \in S,$$

and

$$\lim_{k\to\infty} C_k = 0,$$

then $f_k \to f$ uniformly on S.

In practice, we find $f(x) = \lim_{k\to\infty} f_k(x)$ first (i.e. find pointwise limits of f_k), then prove or disprove (7.1).

As with sequences of vectors we have the notion of Cauchy sequences.

Definition 7.3. A sequence $\{f_k\}$ of functions on S is *uniformly Cauchy* if

$$\forall \varepsilon > 0, \exists K > 0, \forall k, j \in \mathbb{N} : \big[k, j > K \implies (\forall x \in S, |f_k(x) - f_j(x)| < \varepsilon)\big],$$

or equivalently,

$$\forall \varepsilon > 0, \exists K > 0, \forall k, j \in \mathbb{N} : \big[k, j > K \implies \sup\{|f_k(x) - f_j(x)| : x \in S\} < \varepsilon\big].$$

Theorem 7.4. *The sequence $\{f_k\}$ is uniformly convergent on S if and only if it is uniformly Cauchy.*

Theorem 7.5. *Suppose $f_k \to f$ uniformly on S. If each f_k is continuous on S, then so is f.*

Theorem 7.6. *Let $a, b \in \mathbb{R}$ and $a < b$. Suppose f_k (for $k \in \mathbb{N}$) and f are integrable on $[a, b]$ and $f_k \to f$ uniformly on $[a, b]$. Then*

$$\int_a^b \lim_{k\to\infty} f_k(x)dx = \int_a^b f(x)dx = \lim_{k\to\infty} \int_a^b f_k(x)dx.$$

Theorem 7.7. *Let $\{f_k\}$ be a sequence of functions of class C^1 on an interval $[a, b]$. Suppose that $f_k \to f$ pointwise and $f_k' \to g$ uniformly on $[a, b]$. Then f is of class C^1 on $[a, b]$ and $f' = g$.*

7.2 Series of functions

Let $S \subset \mathbb{R}^{m'}$ (m' is used to avoid the conflict with the following index n) and $f_n : S \to \mathbb{R}^m$. Define the partial sum

$$s_k = f_1 + f_2 + \ldots + f_k = \sum_{n=1}^{k} f_n. \tag{7.3}$$

Then $\{s_k\}$ is a sequence of functions and is considered as an infinite series $\sum_{n=1}^{\infty} f_n$.

We say that the series $\sum_1^{\infty} f_n$ is *uniformly convergent* if the sequence of partial sums $\{s_k\}$ is uniformly convergent. The limit of the series is, as usual, the limit of the partial sums.

Note: we can consider (such as for power series) the infinite series of the form $\sum_{n=0}^{\infty} f_n$. Of course, in this case, f_n is defined for all $n \geq 0$.

Theorem 7.8 (Cauchy criterion). *The series $\sum_1^{\infty} f_n$ is uniformly convergent on S if an only if*

$$\begin{aligned}&\forall \varepsilon > 0, \exists K > 0, \forall k, j \in \mathbb{N} :\\ &j > k > K \implies \sup\{|f_{k+1}(x) + f_{k+2}(x) + \ldots + f_j(x)| : x \in S\} < \varepsilon.\end{aligned} \tag{7.4}$$

Corollary 7.9. *If $\sum_1^{\infty} f_n$ is uniformly convergent on S then*

$$\lim_{n\to\infty} \Big(\sup\{|f_n(x)| : x \in S\} \Big) = 0, \tag{7.5}$$

or equivalently, $f_n \to 0$ *uniformly on* S.

Consequently, if f_n *does not converge to zero uniformly on* S *then* $\sum_1^\infty f_n$ *is not uniformly convergent on* S.

Theorem 7.10 (The Weierstrass M-test). *Suppose there are* $M_n \geq 0$ *for* $n \in \mathbb{N}$ *such that*
(i) $|f_n(x)| \leq M_n$, *for all* $n \in \mathbb{N}$ *and* $x \in S$, *and*
(ii) $\sum_1^\infty M_n < \infty$.
Then $\sum_1^\infty f_n$ *is absolutely and uniformly convergent on* S.

Theorem 7.11. *Suppose each* f_n *is continuous on* S *and* $\sum_1^\infty f_n$ *converges to* f *uniformly on* S. *Then* f *is continuous on* S.

Theorem 7.12. *Let* $a, b \in \mathbb{R}$ *and* $a < b$. *Suppose each* f_n *is continuous on* $[a, b]$ *and* $\sum_1^\infty f_n$ *converges to* f *pointwise on* $[a, b]$.
(i) If $\sum_1^\infty f_n$ *converges uniformly on* $[a, b]$ *then*

$$\int_a^b \sum_1^\infty f_k(x)dx = \int_a^b f(x)dx = \sum_1^\infty \int_a^b f_k(x)dx. \tag{7.6}$$

(ii) If each f_k *is of class* C^1 *on* $[a, b]$ *and* $\sum_1^\infty f_k'$ *is uniformly convergent then* $f \in C^1([a, b])$ *and*

$$\frac{d}{dx}\Big[\sum_1^\infty f_n(x)\Big] = f'(x) = \sum_1^\infty f_k'(x) \tag{7.7}$$

7.3 Power series

We consider the infinite series of the form

$$\sum_0^\infty a_n x^n, \quad a_n, x \in \mathbb{R}.$$

This is called a *power series*. Note that when $x = 0$ the series is convergent and its limit is zero.

Theorem 7.13. *For any power series $\sum_0^\infty a_n x^n$, there is a number $R \in [0,\infty]$ such that we have the following.*
(i) The series is absolutely convergent for $|x| < R$.
The series is divergent for $|x| > R$.
The series is uniformly convergent on $[-r, r]$ for any $0 \le r < R$.
(ii) Let $f(x) = \sum_0^\infty a_n x^n$ whenever it is defined. Then
The function f is continuous on $(-R, R)$.
The function f is of class C^1 on $(-R, R)$, and

$$f'(x) = \sum_1^\infty n a_n x^{n-1}, \quad x \in (-R, R). \tag{7.8}$$

For $a, b \in (-R, R)$, the function f is integrable on $[a, b]$ and

$$\int_a^b f(x)dx = \sum_0^\infty \int_a^b a_n x^n dx. \tag{7.9}$$

In particular, for $x \in (-R, R)$, we have

$$\int_0^x f(t)dt = \sum_0^\infty \frac{a_n}{n+1} x^{n+1}. \tag{7.10}$$

Such R above is called *the radius of convergence* of the series. Note that R can be 0 which means the series is divergent for any $x \neq 0$; and R can be ∞ which means the series is absolutely convergent for all $x \in \mathbb{R}$.

R can be defined by

$$R = \sup\{|x| \ : \ \sum_0^\infty a_n x^n \text{ converges}\}. \tag{7.11}$$

How to compute R ? In fact $R = 1/L$ where $L \in [0, \infty]$ is determined by

$$L = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} \text{ or } L = \lim_{n\to\infty} \sqrt[n]{|a_n|}, \tag{7.12}$$

whenever it exists, or in general,

$$L = \limsup_{n\to\infty} \sqrt[n]{|a_n|}. \tag{7.13}$$