Chapter 3

The Implicit Function Theorem and Its Applications

We present the Inverse Mapping Theorem first (Theorem 3.18 in the text) and then the Implicit Function Theorem (Theorem 3.9 in the text)

Theorem 3.1 (The inverse mapping theorem)**.** *Let U and V be open sets in $\mathbb{R}^n$ and $a \in U$. Let $f : U \to V$ be a mapping of class C^1 and $b = f(a)$. Suppose $Df(a)$ is invertible, that is, $\det Df(a) \neq 0$. Then there exist neighborhoods $M \subset U$ of a and $N \subset V$ of b, so that f is a one-to-one map from M onto N, and the inverse map f^{-1} from N to M is also of class C^1. Moreover, if $x \in M$ and $y = f(x) \in N$, then $D(f^{-1})(y) = [Df(x)]^{-1}$.*

Read the example on p. 138 of the textbook.

Let $F : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^k$. For $x \in \mathbb{R}^n, y \in \mathbb{R}^k$, $F(x, y) = (F_1, F_2, \dots, F_k) \in \mathbb{R}^k$. We use the following notation

$$D_x F = \Big(\partial_{x_j} F_i\Big)_{\substack{1 \leq i \leq k \\ 1 \leq j \leq n}} = \begin{pmatrix} \partial_{x_1} F_1 & \partial_{x_2} F_1 & \dots & \partial_{x_n} F_1 \\ \partial_{x_1} F_2 & \partial_{x_2} F_2 & \dots & \partial_{x_n} F_2 \\ \vdots & \vdots & \vdots & \vdots \\ \partial_{x_1} F_k & \partial_{x_2} F_k & \dots & \partial_{x_n} F_k \end{pmatrix},$$

$$D_yF = \left(\partial_{y_j}F_i\right)_{1\le i,j\le k} = \begin{pmatrix} \partial_{y_1}F_1 & \partial_{y_2}F_1 & \dots & \partial_{y_k}F_1 \\ \partial_{y_1}F_2 & \partial_{y_2}F_2 & \dots & \partial_{y_k}F_2 \\ \vdots & \vdots & \vdots & \vdots \\ \partial_{y_1}F_k & \partial_{y_2}F_k & \dots & \partial_{y_k}F_k \end{pmatrix}.$$

Note that D_xF is a $k\times n$ matrix, D_yF is a $k\times k$ matrix and the derivative of F is

$$DF = \begin{pmatrix} D_xF & D_yF \end{pmatrix},$$

a $k\times(n+k)$ matrix.

Theorem 3.2. *Let $U\subset\mathbb{R}^n\times\mathbb{R}^k$ be open and $F: U\to\mathbb{R}^k$ is of class C^1. Let $a\in\mathbb{R}^n$, $b\in\mathbb{R}^k$ such that $(a,b)\in U$. Suppose $F(a,b)=0$ and the matrix $B = D_yF(a,b)$ is invertible, that is, $\det B\neq 0$. Then there are positive numbers r_0 and r_1 such that*

(i) For all $x\in B(r_0,a)$, there exists unique $y\in B(r_1,b)$ such that $(x,y)\in U$ and $F(x,y)=0$.

We define the function $f: B(r_0,a)\to B(r_1,b)$ as follows: for each $x\in B(r_0,a)$, $f(x)$ is that unique $y\in B(r_1,b)$.

(ii) The function f above is of class C^1 and $F(x,f(x))=0$ for all $x\in B(r_0,a)$. Consequently, for $x\in B(r_0,a)$ and $y=f(x)$, we have

$$Df(x) = -[D_yF(x,y)]^{-1}D_xF(x,y),$$

whenever $D_yF(x,y)$ is invertible.

Example 3.3. Consider the problem of solving

$$x - yu^2 = 0, \quad xy + uv = 0 \tag{3.1}$$

for u and v as functions of x and y.

Let $n=k=2$. Set $F=(F_1,F_2)=(x-yu^2, xy+uv)$. We have

$$A = D_{(x,y)}F = \begin{pmatrix} 1 & -u^2 \\ y & x \end{pmatrix},$$

$$B = D_{(u,v)}F = \begin{pmatrix} -2yu & 0 \\ v & u \end{pmatrix}.$$

We have $\det D_{(u,v)}F(x,y,u,v) = -2yu^2$. Therefore, for any solution (x_0, y_0, u_0, v_0) of (3.1) such that $y_0u_0 \neq 0$, we can solve (3.1) for $(u,v) = f(x,y) = (u(x,y), v(x,y))$ nearby the given point (x_0, y_0, u_0, v_0).

For example, let $(x_0, y_0, u_0, v_0) = (1, 1, 1, -1)$ be a solution of (3.1). We want to find also $Df(1,1)$. We have

$$A = D_{(x,y)}F(1,1,1,-1) = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix},$$

$$B = D_{(u,v)}F(1,1,1,-1) = D_{(u,v)}F = \begin{pmatrix} -2 & 0 \\ -1 & 1 \end{pmatrix}.$$

It is known that if $ad - bc \neq 0$, the inverse matrix of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

We have $\det B = -2 \neq 0$ and hence $B^{-1} = \begin{pmatrix} -1/2 & 0 \\ -1/2 & 1 \end{pmatrix}$. Thus

$$\begin{aligned} Df(1,1) = -B^{-1}A &= -\begin{pmatrix} -1/2 & 0 \\ -1/2 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \\ &= -\begin{pmatrix} -1/2 & 1/2 \\ 1/2 & 3/2 \end{pmatrix} = \begin{pmatrix} 1/2 & -1/2 \\ -1/2 & -3/2 \end{pmatrix}. \end{aligned}$$

This implies

$$\partial_x u(1,1) = 1/2,\ \partial_y u(1,1) = -1/2,\ \partial_x v(1,1) = -1/2,\ \partial_y v(1,1) = -3/2.$$