Chapter 1

Setting the stage

1.1 Euclidean spaces and vectors

Let n be a natural number, i.e. $n = 1, 2, 3, \ldots$. The n-dimensional Euclidean space is the set of odered n-tuples of real numbers. We denote this space by $\mathbb{R}^n$. Then

$$\mathbb{R}^n = \{x = (x_1, x_2, \ldots, x_n) : x_1, x_2, \ldots, x_n \in \mathbb{R}\}, \tag{1.1}$$

where $\mathbb{R}$ denotes the set of real numbers. In fact, $\mathbb{R}^n = \mathbb{R} \times \mathbb{R} \times \ldots \times \mathbb{R}$ the Cartesean product of $\mathbb{R}$. Each element in $x = (x_1, x_2, \ldots, x_n)$ is called a vector with components $x_1, x_2, \ldots, x_n$. Other notations for vectors can be bold letters $\mathbf{x}$ or underlined letters $\underline{x}$; however we will not use these in this note.

The zero vector of $\mathbb{R}^n$ is simply $0 = (0, 0, \ldots, 0)$.

Let $x = (x_1, x_2, \ldots, x_n), y = (y_1, y_2, \ldots, y_n)$ be two vectors in $\mathbb{R}^n$ and $c \in \mathbb{R}$. We define the following operations:

Addition: $x + y = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n)$,

Scalar product: $cx = (cx_1, cx_2, \ldots, cx_n)$,

Dot product: $x \cdot y = x_1y_1 + x_2y_2 + \ldots x_ny_n$.

The norm (or the length) of x is

$$|x| = \sqrt{x \cdot x} = \sqrt{x_1^2 + x_2^2 + \ldots + x_n^2}. \tag{1.2}$$

Denote $-x = (-1)x = (-x_1, -x_2, \ldots, -x_n)$.

Some immediate properties:

$$x+y = y+x, (x+y)+z = x+(y+z), c(x+y) = cx+cy, x+(-x) = 0, \quad (1.3)$$

$$|cx| = |c||x|, |-x| = |x|. \quad (1.4)$$

Proposition 1.1 (Cauchy-Schwar's inequality)**.** *For any $a, b \in \mathbb{R}^n$,*

$$|a \cdot b| \leq |a||b|. \quad (1.5)$$

Proof. See text, p.5. ☐

Example 1.2. For $n = 2$, $a = (a_1, a_2)$, $b = (b_1, b_2) \in \mathbb{R}^2$, we have

$$|a_1 b_1 + a_2 b_2| \leq \sqrt{a_1^2 + a_2^2}\sqrt{b_1^2 + b_2^2}. \quad (1.6)$$

For $n = 3$, $a = (a_1, a_2, a_3)$, $b = (b_1, b_2, b_3) \in \mathbb{R}^3$, we have

$$|a_1 b_1 + a_2 b_2 + a_3 b_3| \leq \sqrt{a_1^2 + a_2^2 + a_3^2}\ \sqrt{b_1^2 + b_2^2 + b_3^2}. \quad (1.7)$$

Proposition 1.3 (The triangle inequality)**.** *For any $a, b \in \mathbb{R}^n$,*

$$|a + b| \leq |a| + |b|. \quad (1.8)$$

Consequently,

$$|a - b| \geq |\ |a| - |b|\ |. \quad (1.9)$$

Corollary 1.4. *For any $x, y, z \in \mathbb{R}^n$,*

$$|x - y| \leq |x - z| + |z - y|. \quad (1.10)$$

$$|x| \geq |\ |y| - |x - y|\ |. \quad (1.11)$$

Relation between the norm of x and that of its components: Let $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ and $M = \max\{|x_1|, |x_2|, \ldots, |x_n|\}$, then

$$M \leq |x| \leq \sqrt{n}M. \quad (1.12)$$

1.2 Subsets of Euclidean space

Let $a \in \mathbb{R}^n$ and $r > 0$. The (open) ball $B(r, a)$ is the set of all points whose distance to a is less than r,

$$B(r, a) = \{x \in \mathbb{R}^n : |x - a| < r\}. \tag{1.13}$$

We can also define the closed ball

$$B'(r, a) = \{x \in \mathbb{R}^n : |x - a| \leq r\}. \tag{1.14}$$

Let S be a subset of $\mathbb{R}^n$. Then the complement of S in $\mathbb{R}^n$ is S^c, the set of all points in $\mathbb{R}^n$ that are not in S:

$$S^c = \mathbb{R}^n \setminus S = \{x \in \mathbb{R}^n : x \notin S\}. \tag{1.15}$$

Example 1.5. If $S = B(r, a)$, then $S^c = \{x \in \mathbb{R}^n : |x - a| \geq r\}$. If $S = B'(r, a)$, then $S^c = \{x \in \mathbb{R}^n : |x - a| > r\}$.

Definition 1.6. Let S be a subset of $\mathbb{R}^n$ and $x \in \mathbb{R}^n$.

x is called an interior point of S if there is $r > 0$ such that $B(r, x) \subset S$. We denote the set of interior points of S by S^{int}:

$$S^{\text{int}} = \{x \in \mathbb{R}^n : \exists r > 0, B(r, x) \subset S\}. \tag{1.16}$$

x is called a boundary point of S every ball centered at x intersect both s and S^c, i.e.,

$$\forall r > 0, B(r, x) \cap S \neq \emptyset \text{ and } B(r, x) \cap S^c \neq \emptyset. \tag{1.17}$$

We denote by ∂S the set of all boundary points of S called the boundary of S:

$$\partial S = \{x \in \mathbb{R}^n : \forall r > 0, B(r, x) \cap S \neq \emptyset \text{ and } B(r, x) \cap S^c \neq \emptyset\}. \tag{1.18}$$

The closure of S is $\bar{S} = S \cup \partial S$.

S is a neighborhood of x if x is an interior point of S.

Definition 1.7. Let S be a subset of $\mathbb{R}^n$.

S is called open if it contains none of its boundary points: $S \cap \partial S = \emptyset$.

S is called closed if it contains all of its boundary points: $\partial S \subset S$.

Note: $\mathbb{R}^n$ and the empty set $\emptyset$ are both open and closed.

Two sets A and B are said to be disjoint if $A \cap B = \emptyset$.

Proposition 1.8. *Let S be a subset of $\mathbb{R}^n$. Then*

a. S and its complement S^c have the same boundary: $\partial S = \partial(S^c)$.

b. $S^{\text{int}}, \partial S, (S^c)^{\text{int}}$ are mutually disjoint, i.e., $S^{\text{int}} \cap \partial S, (S^c)^{\text{int}} \cap S^{\text{int}}, \partial S \cap (S^c)^{\text{int}}$ are empty sets.

c. $\mathbb{R}^n = S^{\text{int}} \cup \partial S \cup (S^c)^{\text{int}}$.

Consequently, every point $x \in \mathbb{R}^n$ belongs to exactly one of the following sets $S^{\text{int}}, \partial S, (S^c)^{\text{int}}$.

We also have $S \subset S^{\text{int}} \cup \partial S$, hence $\bar{S} = S^{\text{int}} \cup \partial S$, therefore

Proposition 1.9. $(\bar{S})^c = (S^c)^{\text{int}}$.

Proposition 1.10. *Suppose $S \subset \mathbb{R}^n$.*

a. S is open $\iff$ every point of S is an interior point of S $\iff$ $S = S^{\text{int}}$.

b. S is closed $\iff$ S^c is open.

Proposition 1.11. *(i) If S_1 and S_2 are both open (or closed), so are $S_1 \cup S_2$ and $S_1 \cap S_2$.*

(ii) If $\{S_\alpha\}_{\alpha \in I}$ is a fimily of open sets, then $\cup_{\alpha \in I} S_\alpha$ is open.

(iii) If $\{S_\alpha\}_{\alpha \in I}$ is a fimily of closed sets, then $\cap_{\alpha \in I} S_\alpha$ is closed.

1.3 Limits and continuity

Let n and k be two natural numbers. Let f be a function form $\mathbb{R}^n$ to $\mathbb{R}^k$, $a \in \mathbb{R}^n$ and $L \in \mathbb{R}^k$. We say the limit of $f(x)$ as x aproaches a is L if

$$\forall \varepsilon > 0, \exists \delta > 0, \forall x \in \mathbb{R}^n : 0 < |x - a| < \delta \implies |f(x) - L| < \varepsilon. \tag{1.19}$$

Notation:

$$\lim_{x \to a} f(x) = L. \tag{1.20}$$

Proposition 1.12. *The limit* $\lim_{x \to a} f(x)$*, if exists, is unique.*

Some equivalent statements of (1.19):

- If $a = (a_1, a_2, \dots, a_n)$, then we have $\lim_{x \to a} f(x) = L$ if and only if

$$\begin{aligned} &\forall \varepsilon > 0, \exists \delta > 0, \forall x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n : \\ &0 < |x - a| < \max\{|x_1 - a_1|, |x_2 - a_2|, \dots, |x_n - a_n|\} < \delta \implies |f(x) - L| < \varepsilon. \end{aligned} \tag{1.21}$$

- If $f = (f_1, f_2, \dots, f_k)$ and $L = (L_1, L_2, \dots, L_k)$, where each f_j is a function from $\mathbb{R}^n$ to $\mathbb{R}$ then

$$\lim_{x \to a} f(x) = L \iff \lim_{x \to a} f_j(x) = L_j \text{ for all } j = 1, 2, \dots, k. \tag{1.22}$$

Example 1.13. See text, p. 14, 15.

Proposition 1.14. *Let* $f, g : \mathbb{R}^n \to \mathbb{R}^m$, $a \in \mathbb{R}^n$ *and*

$$\lim_{x \to a} f(x) = L, \quad \lim_{x \to a} g(x) = K. \tag{1.23}$$

Then

(i) $\lim_{x \to a} (f + g)(x) = L + K$.

In the case $m = 1$*, we have*

(ii) $\lim_{x \to a} (fg)(x) = LK$.

(iii) If $L \neq 0$*, then*

$$\lim_{x \to a} \frac{g(x)}{f(x)} = \frac{K}{L}.$$

Remark 1.15. We have

$$\lim_{x\to a} = L \text{ if and only if } \lim_{x\to a} |f(x) - L| = 0. \tag{1.24}$$

When $L = 0$, it becomes

$$\lim_{x\to a} f(x) = 0 \text{ if and only if } \lim_{x\to a} |f(x)| = 0. \tag{1.25}$$

Proposition 1.16 ("squeezing property")**.** *Let $f, g, h : \mathbb{R}^n \to \mathbb{R}$ satisfying $g(x) \le f(x) \le h(x)$ for all $x \in \mathbb{R}^n$. Suppose $a \in \mathbb{R}^n$ and*

$$\lim_{x\to a} g(x) = \lim_{x\to a} h(x) = L \in \mathbb{R}^m.$$

Then $\lim_{x\to a} f(x) = L$.

Proposition 1.17. *Let $f : \mathbb{R}^n \to \mathbb{R}$, $a \in \mathbb{R}^n$ and $\lim_{x\to a} f(x) = L$.*

(i) If $f(x) \le M$ for all $x \in B(r, a)$ for some $r > 0$ then $L \le M$.

(ii) If $f(x) \ge m$ for all $x \in B(r, a)$ for some $r > 0$ then $L \ge m$.

Definition 1.18. Let $a \in \mathbb{R}^n$, we say f is continuous at a if

$$\lim_{x\to a} f(x) = f(a), \tag{1.26}$$

equivalently,

$$\forall \varepsilon > 0, \exists \delta > 0, \forall x \in \mathbb{R}^n : |x - a| < \delta \implies |f(x) - L| < \varepsilon. \tag{1.27}$$

Let U be a subset of $\mathbb{R}^n$. We say f is continuous on U if f is continuous at every point a of U.

Proposition 1.19. *Let $U \subset \mathbb{R}^n$ and $f, g : \mathbb{R}^n \to \mathbb{R}^m$ be continuous on U. Then $(f + g)$ and $(f \cdot g)$ are continuous on U.*

In the case $m = 1$, we have (fg) is continuous on U and (f/g) is continuous on $V = U \setminus g^{-1}(\{0\}) = \{x \in U : g(x) \neq 0\}$.

Theorem 1.20. *Let $f : \mathbb{R}^n \to \mathbb{R}^k$, $g : \mathbb{R}^k \to \mathbb{R}^m$, and $U \subset \mathbb{R}^n$. If f is continuous on U and g is continuous on $f(U)$ then $g \circ f$ is continuous on U.*

Theorem 1.21. *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be continuous and U be a subset of $\mathbb{R}^m$. If U is open (resp. closed), then $f^{-1}(U)$ is open (resp. closed).*

1.4 Sequences

Let A be a non-empty set. A sequence in A is a function $f : \mathbb{N} \to A$, that is, for all $k \in \mathbb{N}$, $x_k = f(k) \in A$. Notation $\{x_k\}, \{x_k\}_1^\infty, \{x_k\}_{k=1}^\infty, \ldots$.

Definition 1.22. Let $\{x_k\}$ be a sequence in $\mathbb{R}^n$ and $L \in \mathbb{R}^n$. We say $\{x_k\}$ coverges to the limit L if

$$\forall \varepsilon > 0, \exists K \in \mathbb{N}, \forall k \in \mathbb{N} : k > K \implies |x_k - L| < \varepsilon. \tag{1.28}$$

Notation:

$$\lim_{k\to\infty} x_k = L.$$

In this case, we say the sequence is *convergent*, otherwise the sequence is *divergent*.

In the case $m = 1$ we have the following two definitions

$$\lim_{k\to\infty} x_k = \infty \iff \forall M > 0, \exists K \in \mathbb{N}, \forall k \in \mathbb{N} : k > K \implies x_k > M, \tag{1.29}$$

$$\lim_{k\to\infty} x_k = -\infty \iff \forall M > 0, \exists K \in \mathbb{N}, \forall k \in \mathbb{N} : k > K \implies x_k < -M. \tag{1.30}$$

If $\lim_{k\to\infty} x_k = \infty$ or $-\infty$ then $\{x_k\}$ is divergent.

Limits of sequences have similar properties to those of limits of functions.

Theorem 1.23. *Suppose $S \subset \mathbb{R}^n$ and $x \in \mathbb{R}^n$. Then x belongs to the closure of S if and only if there is a sequence in S coverging to x.*

Corollary 1.24. *Let S be a subset of $\mathbb{R}^n$. Then S is closed if and only if for every sequence $\{x_k\}$ in S which converges to $a \in \mathbb{R}^n$, we have $a \in S$.*

Theorem 1.25. *Let $S \subset \mathbb{R}^n$, $f : S \to \mathbb{R}^m$ and $a \in S$. Then the following are equivalent*

a. f is continuous at a.

b. For any sequence $\{x_k\}$ in S that converges to a, the sequence $\{f(x_k)\}$ converges to $f(a)$.

Let $\{x_k\}_{k=1}^{\infty}$ be a sequence. Let k_j be a strictly increasing function from $\mathbb{N}$ to $\mathbb{N}$, that is, $k_j \in \mathbb{N}$ for all $j \in \mathbb{N}$ and $k_j > k_l$ whenever $j > l$. Note that the latter property is equivalent to $k_{j+1} > k_j$ for all $j \in \mathbb{N}$. Then the sequence $\{x_{k_j}\}_{j=1}^{\infty}$ is called a *subsequence* of $\{x_k\}$.

Lemma 1.26. *Let k_j be a strictly increasing function from $\mathbb{N}$ to $\mathbb{N}$. Then $k_j \geq j$ for all $j \in \mathbb{N}$.*

Proposition 1.27. *Let $\{x_k\}_{k=1}^{\infty}$ be a convergent sequence in $\mathbb{R}^n$. Then any subsequence $\{x_{k_j}\}_{j=1}^{\infty}$ of $\{x_k\}$ is convergent and*

$$\lim_{j\to\infty} x_{k_j} = \lim_{k\to\infty} x_k.$$

1.5 Completeness

Let $S \subset \mathbb{R}$ and $c \in \mathbb{R}$.

- c is an *upper bound* of S if $\forall x \in S, x \leq c$.
- S is said to be *bounded (from) above* if it has an upper bound.
- c is a *lower bound* of S if $\forall x \in S, x \geq c$.
- S is said to be *bounded (from) below* if it has an lower bound.
- We say S is *bounded* if it is bounded above and below, equivalently there are $m, M \in \mathbb{R}$ such that $m \leq x \leq M$ for all $x \in S$, or equivalently, there is $C > 0$ such that $|x| \leq C$ for all $x \in S$.
- A *least upper bound* of S, called $\sup S$, is an upper bound of S and is smallest among the all upper bounds of S.
- A *greatest lower bound* of S, called $\inf S$, is a lower bound of S and is largest among the all lower bounds of S.

Note that if $\sup S$ (or $\inf S$) exists then it is unique.
Let $A \subset B \subset \mathbb{R}$. Then

$$\sup A \leq \sup B, \quad \inf B \leq \inf A. \tag{1.31}$$

Let $A \subset \mathbb{R}$. Let $B = \{-x : x \in A\}$. If $\sup A$ (resp. $\inf A$) exists then

$$\inf B = -\sup A \quad (\text{resp. } \sup B = -\inf A). \tag{1.32}$$

Proposition 1.28. *Let $S \subset \mathbb{R}$. Then*

$$a = \sup S \iff \begin{cases} (i)\ \forall x \in S, x \leq a, \\ (ii)\ \forall \varepsilon > 0, \exists x_0 \in S : a - \varepsilon < x_0. \end{cases}$$

$$a = \inf S \iff \begin{cases} (i)\ \forall x \in S, x \geq a, \\ (ii)\ \forall \varepsilon > 0, \exists x_0 \in S : x_0 < a + \varepsilon. \end{cases}$$

Remark 1.29. From Proposition 1.28 we see that if $a = \sup S$ or $a = \inf S$ then there is a sequence in S converging to a.

The Completeness Axiom. Let S be a non-empty subset of $\mathbb{R}$ which is bounded above, then $\sup S$ exists.

Corollary 1.30. *Let S be a non-empty subset of $\mathbb{R}$ which is bounded below, then $\inf S$ exists.*

Definition 1.31. Let $\{x_k\}$ be a sequence in $\mathbb{R}$.

- $\{x_k\}$ is *increasing* if $x_k \geq x_j$ whenever $k > j$, or equivalently, $x_{k+1} \geq x_k$ for all k.
- $\{x_k\}$ is *decreasing* if $x_k \leq x_j$ whenever $k > j$, or equivalently, $x_{k+1} \leq x_k$ for all k.
- $\{x_k\}$ is *monotone* if it is increasing or decreasing.
- $\{x_k\}$ is *bounded above* if the set $\{x_k : k \in \mathbb{N}\}$ is bounded above, that is, there is $M \in \mathbb{R}$ such that $x_k \leq M$ for all k.
- $\{x_k\}$ is *bounded below* if the set $\{x_k : k \in \mathbb{N}\}$ is bounded below, that is, there is $m \in \mathbb{R}$ such that $x_k \geq m$ for all k.
- $\{x_k\}$ if *bounded* if it is bounded above and below, equivalently, there is $C > 0$ such that $|x_k| < C$ for all k.

Theorem 1.32. *Every bounded monotone sequence in $\mathbb{R}$ is convergent. More precisely,*

(i) If $\{x_k\}$ is increasing and bounded above then

$$\lim_{k\to\infty} x_k = \sup\{x_k : k \in \mathbb{N}\}. \tag{1.33}$$

(ii) If $\{x_k\}$ is decreasing and bounded below then

$$\lim_{k\to\infty} x_k = \inf\{x_k : k \in \mathbb{N}\}. \tag{1.34}$$

Theorem 1.33 (The nested interval theorem)**.** *Let* $I_k = [a_k, b_k]$ *for* $k \in \mathbb{N}$, $a_k, b_k \in \mathbb{R}$, $a_k \leq b_k$, *be a sequence of intervals that satisfy*

(a) $I_1 \supset I_2 \supset I_3 \supset \ldots$, *that is,* $I_k \supset I_{k+1}$ *for all* k.

(b) $\lim_{k\to\infty}(b_k - a_k) = 0$.

Then $\cap_{k=1}^{\infty} I_k = \{c\}$ *for some* $c \in \mathbb{R}$.

Using the nested interval theorem, we can prove

Theorem 1.34. *Every bounded sequence in* $\mathbb{R}$ *has a convergent subsequence.*

As a consequence, we have

Theorem 1.35. *Every bounded sequence in* $\mathbb{R}^n$ *has a convergent subsequence.*

Proposition 1.36. *Let* $\{x_k\}$ *be a convergent sequence in* $\mathbb{R}^n$. *Then*

(a) $\{x_k\}$ *is bounded.*

(b) roughly speaking, $(x_k - x_j) \to 0$ *as* $k, j \to \infty$; *more precisely,*

$$\forall \varepsilon > 0, \exists K \in \mathbb{N}, \forall k \in \mathbb{N}, \forall j \in \mathbb{N} : [(k > K) \wedge (j > K)] \implies |x_k - x_j| < \varepsilon. \tag{1.35}$$

Definition 1.37. A sequence in $\mathbb{R}^n$ is called a *Cauchy sequence* if it satisfies (1.35).

Proposition 1.38. *Let* $\{x_k\}$ *be a Cauchy sequence in* $\mathbb{R}^n$. *Then it is bounded. If, in addition, it has a convergent subsequence* $\{x_{k_j}\}_{j=1}^{\infty}$ *then* $\{x_k\}$ *itseft is convergent and* $\lim_{k\to\infty} x_k = \lim_{j\to\infty} x_{k_j}$.

Combining Theorem 1.35, Propositions 1.36 and 1.38, we obtain

Theorem 1.39. *A sequence in* $\mathbb{R}^n$ *is convergent if and only if it is Cauchy.*

1.6 Compactness

Definition 1.40. A subset in $\mathbb{R}^n$ is called *compact* if it is closed and bounded.

Theorem 1.41 (The Bozano-Weierstrass Theorem)**.** *Let S ba a subset of $\mathbb{R}^n$. Then the following are equivalent*

(a) S is compact

(b) Every sequence in S has a subsequence converging to a point which belongs to S.

The raltion between compact sets and continuous functions:

Theorem 1.42. *Let $S \subset \mathbb{R}^n$ be compact and $f : S \to \mathbb{R}^m$ be continuous. Then $f(S)$ is compact (as a subset of $\mathbb{R}^m$).*

Corollary 1.43. *Let $S \subset \mathbb{R}^n$ be compact and $f : S \to \mathbb{R}^m$ be continuous.*

Definition 1.44. Let $S \subset \mathbb{R}^n$, $f : S \to \mathbb{R}$, and $a \in S$.

$f(a)$ is the *maximum* (largest value) of f on S if $f(a) \geq f(x)$ for all $x \in S$.

$f(a)$ is the *minimum* (smallest value) of f on S if $f(a) \leq f(x)$ for all $x \in S$.

Theorem 1.45 (The Extreme Value Theorem)**.** *Let $S \subset \mathbb{R}^n$ be compact and $f : S \to \mathbb{R}^m$ be continuous. Then there are $a, b \in S$ such that $f(a)$ is the maximum value of f on S and $f(b)$ is the minimum value of f on S.*

1.7 Connectedness

Let S be a subset of $\mathbb{R}^n$.

- S is *disconnected* if there are non-empty sets S_1 and S_2 such that

$$S = S_1 \cup S_2, \quad S_1 \cap \bar{S}_2 = \emptyset, \quad S_2 \cap \bar{S}_1 = \emptyset. \tag{1.36}$$

  We call the above pair (S_1, S_2) a *disconnection* of S. (Note: they are not unique.)

- S is *connected* if it is NOT disconnected.

Theorem 1.46. *The connected subsets of* $\mathbb{R}$ *are the intervals, i.e.,* $[a, b), [a, b], (a, b], (a, b), [c, \infty), (c, \infty), (-\infty, c), (-\infty, c]$.

Proof. Skipped (see text). $\square$

Notes: S is an interval in $\mathbb{R}$ if and only if

$$\forall x, y \in S, \forall z \in \mathbb{R} : x < z < y \implies z \in S. \tag{1.37}$$

Theorem 1.47. *If* $S \subset \mathbb{R}^n$ *is connected and* $f : S \to \mathbb{R}^m$ *is continuous, then* $f(S)$ *is connected.*

Proof. Proof by Contraposition: $f(S)$ being disconnected implies S being disconnected.

Suppose $f(S)$ is disconnected then it has a disconnection (U_1, U_2). Let $S_1 = f^{-1}(U_1) = \{x \in S : f(x) \in U_1\}$ and $S_2 = f^{-1}(U_2) = \{x \in S : f(x) \in U_1\}$. Then S_1, S_2 are not empty and $S_1 \cup S_2 = S$. Suppose $S_1 \cap \bar{S}_2 \neq \emptyset$, then there is $x_0 \in S_1 \cap \bar{S}_2$. There is a sequence $\{x_k\}$ in S_2 such that $x_k \in S_2$, $x_k \to x_0$ as $k \to \infty$. Since f is continuous at $x_0 \in S$: $\lim_{k\to\infty} f(x_k) = f(x_0)$. Note that $f(x_k) \in U_2$, then $f(x_0) \in \bar{U}_2$. But we also have $x_0 \in S_1$ which implies $f(x_0) \in U_1$, therefore $f(x_0) \in U_1 \cap \bar{U}_2$. This contradicts the fact that $U_1 \cap \bar{U}_2 = \emptyset$. Thus $S_1 \cap \bar{S}_2 = \emptyset$. Similarly, $S_2 \cap \bar{S}_1 = \emptyset$. Hence S is disconnected. $\square$

Corollary 1.48 (The intermediate value theorem)**.** *Suppose S is connected and $f : S \to \mathbb{R}$ is continuous. If $a, b \in S$, $t \in \mathbb{R}$ and $f(a) < t < f(b)$, then there is $c \in S$ such that $f(c) = t$.*

Proof. We have $f(S)$ is a connected subset of $\mathbb{R}$, hence it is an interval. Since $f(a), f(b) \in f(S)$, then we have the whole interval $[f(a), f(b)]$ is contained in $f(S)$. Therefore $t \in f(S)$, which means that there is $c \in S$ such that $t = f(c)$. □

Definition 1.49. A set $S \subset \mathbb{R}^n$ is said to be *arcwise connected* (or *pathwise connected*) if any two points in S can be joined by a continuous curve in S, that is for any $a, y \in S$, there is a continuous function $g : [0, 1] \to S$ such that $g(0) = a$ and $g(1) = b$.

Theorem 1.50. *If S is arcwise connected, then S is connected.*

Proof. Let S be arcwise connected. Suppose S is disconnected. Let (S_1, S_2) be a disconnection of S. There are $a \in S_1$ and $b \in S_2$. Since S is arcwise connected there is a continuous function $f : [0, 1] \to S$ such that $f(0) = a$ and $f(1) = b$. Note that $T = f([0, 1])$ is connected. Let $T_1 = S_1 \cap T$ and $T_2 = S_2 \cap T$. Then T_1, T_2 are non-empty sets (containing a, b respectively.). We have $T_1 \cap \bar{T}_2 \subset S_1 \cap \bar{S}_2 = \emptyset$, hence $T_1 \cap \bar{T}_2 = \emptyset$. Similarly, $T_2 \cap \bar{T}_1 = \emptyset$. Therefore, T is disconnected, contradiction. Conclusion: S is connected. □

Let $a, b, c \in S$. If there is a countinuous curve in S connecting a and b, and one connectiong b and c, then there is one connecting a and c (transitive relation). Indeed, let $f, g : [0, 1] \to S$ such that $f(0) = a, f(1) = b$ and $g(0) = b, g(1) = c$. Then let $h : [0, 1] \to S$,

$$h(t) = \begin{cases} f(2t) & \text{if } 0 \leq t < 1/2, \\ g(21 - 1) & \text{if } 1/2 \leq t \leq 1. \end{cases}$$

(Verify the continuity of h at $1/2$ using left and right limits.)

Example 1.51. Balls, spheres in $\mathbb{R}^3$ and disks, circles in $\mathbb{R}^2$ are arcwise-connected, hence connected.

Example 1 p.34 in the text. In $\mathbb{R}^2$, let $a = (-1,0), b = (1,0)$ and $S_1 = B(1,a), S_2 = B(1,b)$. Let $S = S_1 \cup S_2$ and $T = S_1 \cap \bar{S}_2$. Then S is disconnected. Since every point in T can be connected to the origin $(0,0) \in T$, we have T is arcwise connected, hence connected.

Note: A connected set is not necessarily arcwise connected. See text p.37 for an example of a set in $\mathbb{R}^2$ which is connected but NOT arcwise-connected.

Theorem 1.52. *If S is connected and open, then S is arcwise connected.*

Proof. Let S be open and connected. Let a be a fixed point in S. We will prove that we can connect a to any other points of S, hence showing that S is arcwise connected.

Set $S_1 = \{x \in S : x$ is joined by a continuous curve in $S\}$.
Claim: $S_1 = S$. Then S is arcwise connected.

Proof of the claim: Suppose $S_1 \neq S$. Then $S_2 = S \setminus S_1$ is not empty and $S = S_1 \cup S_2$. Note: $S_1 \neq \emptyset$ and $S_1 \cap S_2 = \emptyset$. We now show that $S_1 \cap \bar{S}_2$ and $S_2 \cap \bar{S}_1$ are empty.

Let $x \in S_1$, S being open implies there is a ball $B(r,x) \subset S$, $r > 0$. For every $y \in B$, there is a curve from a to x then x to y, hence $y \in S_1$. Therefore $B(1,x)$ is a subset of S_1. Thus $x \notin \bar{S}_2$. We then have $S_1 \cap \bar{S}_2 = \emptyset$.

Let $x \in S_2$, there is a ball $B = B(r,x) \subset S$. Suppose $x \in \bar{S}_1$ then there is $y \in B \cap S_1$, hence we can find a continuous curve in S from a to y then y to x. Thus $x \in S_1$, which is absurd since $x \notin S_1$ $(S_1 \cap S_2 = \emptyset)$. Hence $x \notin \bar{S}_1$, therefore $S_2 \cap \bar{S}_1 = \emptyset$.

We have proved (S_1, S_2) is a disconnection of S, which is impossible since S is connected. Therefore the claim is true and the proof of the theorem is complete. $\square$

1.8 Uniform continuity

Let $S \subset \mathbb{R}^n$ and $f : S \to \mathbb{R}^m$ be continuous. We have

$$\forall x \in S, \forall \varepsilon > 0, \exists \delta > 0, \forall y \in S : |y - x| < \delta \implies |f(x) - f(y)| < \varepsilon. \quad (1.38)$$

The above δ in general depends on x, ε. In some cases, δ is independent of x, then roughly speaking, the rate $f(y)$ approaches $f(x)$ as y approaches x is controlled uniformly on the whole domain S.

Definition 1.53. A function $f : S \to \mathbb{R}^m$ is *uniformly continuous* on S if

$$\forall \varepsilon > 0, \exists \delta > 0, \forall x \in S, \forall y \in S : |y - x| < \delta \implies |f(x) - f(y)| < \varepsilon. \quad (1.39)$$

Example 1.54. The function $f(x) = x^2$ is not uniformly continuous on $(0, \infty)$. Suppose it is, let $\varepsilon > 0$, then there is $\delta > 0$ such that for any $x, y \in (0, \infty)$ and $\delta > 0$, we have

$$|y^2 - x^2| = |y - x||y + x| < \varepsilon.$$

Take $y = x + \delta$ then $2\delta x < \varepsilon$. So $\delta < \varepsilon/(2x)$ which goes to zero as x goes to infinity which is a contradiction since δ is a fixed positive number.

Example 1.55. The function $f(x) = \sin x$ is uniformly continuous on $\mathbb{R}$. Indeed, by the Mean Value Theorem (next chapter), $|f(x) - f(y)| = |x - y||\cos z| \le |x - y|$, where $z \in [x, y]$ or $[y, x]$. We can take $\delta = \varepsilon$ in (1.39).

Example 1.56. The function $f(x) = x^2$ is uniformly continuous on every bounded subsets of $\mathbb{R}$. Suppose there is $M > 0$ such that $|x| \le M$ for all $x \in S$. Then for any $x, y \in S$.

$$|f(x) - f(y)| = |x - y||x + y| \le 2M|x - y|.$$

We can take $\delta = \varepsilon/(2M)$ in (1.39). *Note:* We can use the Mean Value Theorem as well.

Theorem 1.57. *Suppose S is compact and $f : S \to \mathbb{R}^m$ is continuous. Then f is uniformly continuous.*

Proof. By contradiction. Suppose f is not uniformly continuous, then

$$\exists \varepsilon_0 > 0, \forall \delta > 0, \exists x, y \in S : |x - y| < \delta \text{ and } |f(x) - f(y)| \geq \varepsilon_0. \tag{1.40}$$

Take $\delta = 1/k \to 0$. There are sequences $\{x_k\}, \{y_k\}$ in S such that

$$|x_k - y_k| < \frac{1}{k}, \quad |f(x_k) - f(y_k)| \geq \varepsilon_0. \tag{1.41}$$

Since S is compact, there exist covergent subsequences $\{x_{k_j}\}, \{y_{k_j}\}$ whose limits belong to S. By the first property of (1.41), we have

$$\lim_{j\to\infty} x_{k_j} = \lim_{j\to\infty} y_{k_j} = x_0 \in S.$$

Since f is continuous at x_0, $\lim_{j\to\infty} |f(x_{k_j}) - f(y_{k_j})| = |f(x_0) - f(x_0)| = 0$ which contradicts the second property in (1.41). We conclude that f must be uniformly continuous. $\square$