Research Statement

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My research interests are partial differential equations (PDE), fluid dynamics and dynamical systems. Particularly, my works are focused on the following topics:

(i) Dynamics of viscous, incompressible fluids related to the Navier–Stokes equations (NSE).
(ii) Generalized Forchheimer (non-Darcy) flows of compressible fluids in porous media.
(iii) Asymptotic analysis for dynamical systems.

The first two subjects are quite different. They both deal with fundamental and challenging nonlinear PDE in fluid dynamics, but require distinct techniques. Their study requires both insights and innovation. The last subject consists of analysis for both ordinary differential equations (ODE) and abstract dynamical systems, which can be applied to PDE.

1. Dynamics of viscous, incompressible fluids

My long-term interest is the long-time dynamics of the fluid flows with the ultimate goal being to understand the fluid turbulence. Fluid turbulence, of course, is one of the most, if not the most, challenging problems in classical physics. However, much is known about the decaying turbulence with potential or non-potential body forces, which can help shed insights into the other types of turbulence. This is part of my contribution to the subject. My current research in this direction contains two subcategories that correspond to two descriptions of fluids: the Eulerian and the Lagrangian.

1.1. **The Navier–Stokes equations.**

The NSE is a system of nonlinear PDEs that describe the dynamics of the incompressible, viscous fluid flows. Although the research on them is vast, the fundamental questions about the three-dimensional NSE are still open.

My strength in this direction lies in the asymptotic expansions for the solutions of the three-dimensional (3D) NSE in different contexts. I have published extensively on the subject and am continuing to expand the theory. Based on the pioneering works by Foias and Saut [24–28], my collaborators and I developed this theory further in [19–23, 48, 52]. We have detailed asymptotic analysis of solutions of the NSE: asymptotic expansions of solutions, normalization map and normal form theory. We develop analytic tools for studying dynamical systems in infinite dimensional spaces, particularly infinite integrable systems, the normal form theory and Poincaré-Dulac normal forms. We apply them to the mathematical theory of decaying turbulence based on the NSE with potential body forces. It includes asymptotic analysis of statistical solutions, ensemble averages of physical quantities and relations between them, and connections to Kolmogorov's theory on turbulence. The current development is the asymptotic theory for the NSE with non-potential, decaying forces [3,4,38,49]. It is very promising in deriving a rigorous theory of decaying turbulence, at least in certain interesting, but still rather general, scenarios. Our methods and techniques invented for the NSE can be easily adapted to other dissipative dynamical systems.

We consider the 3D NSE for for viscous, incompressible fluids

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} - \nu\Delta\mathbf{u} = -\nabla p + \mathbf{f}, \quad \text{div } \mathbf{u} = 0, \tag{1.1}$$

where $\nu > 0$ is the kinematic viscosity, $\mathbf{u}$ is the velocity field, p is the pressure, and $\mathbf{f}$ is the body force. We study the case of periodic boundary conditions with the domain of periodicity $\Omega = (-L/2, L/2)^3$. The force $\mathbf{f}$ is assumed to be divergence-free and has zero spatial average over Ω. Let H be the closure in $L^2(\Omega)^3$ of $\mathcal{V}$ – the set of all vector-valued L-periodic trigonometric polynomials which are divergence-free and have zero spatial average. Denote $u(t) = \mathbf{u}(\cdot, t)$, $f(t) = \mathbf{f}(\cdot, t)$ and the Stokes operator $A = -\Delta$.

1.1.1. *Potential body forces.* When $f = 0$, it is proved in [27, 28] that for any Leray–Hopf weak solution $u(t)$ of (1.1) has the asymptotic expansion

$$u(t) \sim \sum_{n=1}^{\infty} q_n(t) e^{-\mu_n t}, \tag{1.2}$$

where $0 < \mu_n \to \infty$ as $n \to \infty$, each $q_n(t)$ is a polynomial in t with values in $\mathcal{V}$. This means that for any $N \in \mathbb{N}$ the correction term $\tilde{u}_{N+1}(t) = u(t) - \sum_{j=1}^{N} q_j(t)e^{-jt}$ satisfies

$$\|\tilde{u}_{N+1}(t)\|_{L^2(\Omega)} = O\big(e^{-(N+\varepsilon)t}\big) \quad \text{as} \quad t \to \infty \quad \text{for some} \quad \varepsilon = \varepsilon_N > 0. \tag{1.3}$$

In fact, for each $m \in \mathbb{N}$ relation (1.3) holds for the Sobolev norm $\|\tilde{u}_{N+1}(t)\|_{H^m(\Omega)}$ and $\varepsilon = \varepsilon_{N,m} > 0$.

Associated to the expansion (1.2) is a normalization map $\xi = W(u_0)$, with u_0 being an initial data. For a solution $u(t)$, the function $\xi(t) = W(u(t)$ satisfies an equation

$$\xi' + A\xi + \sum_{d=2}^{\infty} \mathcal{B}^{[d]}(\xi) = 0, \tag{1.4}$$

where $\mathcal{B}^{[d]}$ is a homogeneous polynomials in ξ of degree d. Foias, Saut and I prove in [23] that equation (1.4), in fact, is a Poincaré–Dulac normal form.

In [52], Titi and I study the NSE for the rotating fluids, which are

$$\frac{\partial \mathbf{u}}{\partial t} - \nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p + \Omega\mathbf{e}_3 \times \mathbf{u} = 0, \quad \operatorname{div}\mathbf{u} = 0, \tag{1.5}$$

where $\frac{1}{2}\Omega\mathbf{e}_3$ is the angular velocity of the rotation.

Theorem 1.1 (Hoang–Titi [52])**.** *Let $u(t)$ be any Leray-Hopf weak solution of (1.5). Then there exist $\mathcal{V}$-valued SS-polynomials $\mathcal{Q}_n(t)$'s, for all $n \geq 0$, such that*

$$u(t) \sim \mathcal{Q}_0(t) + \sum_{n=1}^{\infty} \mathcal{Q}_n(t)e^{-\mu_n t}. \tag{1.6}$$

Above, a function $g : \mathbb{R} \to X$ is an X-valued SS-polynomial if it is a finite sum of the functions of the forms $t^m \cos\big(a\cos(\omega t) + b\sin(\omega t) + ct + d\big)Z$, $t^m \sin\big(a\cos(\omega t) + b\sin(\omega t) + ct + d\big)Z$, with $m \in \mathbb{N} \cup \{0\}, a, b, c, d, \omega \in \mathbb{R}, Z \in X$. Note that the asymptotic expansion (1.6) with "double sinusoidal" functions is a new phenomenon, which was not observed before.

1.1.2. *Non-potential body forces.* The asymptotic expansions for solution $u(t)$ corresponding to decaying force $f(t)$ are investigated in different contexts in [3,4,38,49]. The general form of results is the following.

Theorem 1.2. *If $f(t)$ has an asymptotic expansion $f(t) \sim \sum_{k=1}^{\infty} f_k(t)$, then any decaying solution $u(t)$ has a corresponding asymptotic expansion $u(t) \sim \sum_{k=1}^{\infty} u_k(t)$.*

The class of functions $f_k(t)$ and $u_k(t)$ can be very general and complicated. For example, in [38], it contains not only the power functions but also the sinusoidal ones of t, $\ln t$, $\ln\ln t$, etc.

1.1.3. *The Garlerkin approximation with large Grashof number.* In the recent paper [18], Foias, Jolly and myself study the Garlerkin approximation of the NSE with large Grashof number G. After appropriate scaling, we consider the stationary equation

$$Av_n + G_n B(v_n, v_n) = g$$

in a finite dimensional space with the Grashof number $G_n \to \infty$ as $n \to \infty$. We obtain a new asymptotic expansion, as $n \to \infty$,

$$v_n \sim v + \sum_{k=1}^{\infty} \Gamma_{n,k} w_k, \quad |w_k| = 1, \quad \lim_{n\to\infty} \Gamma_{n,k+1}/\Gamma_{n,k} = 0.$$

Many relations between v and w_k are discovered. The methods in this paper may provide an alternative approach to studying turbulence, especially with numerical computations.

1.2. **Analysis of the Lagrangian trajectories.** The Lagrangian formulation of fluids results in more complicated equations than the NSE which come from the Eulerian formulation. Therefore, the analysis of their solutions is very limited, and often restricted to short time properties. Yet in a recent paper [32], I managed to obtain the asymptotic expansions, as $t \to \infty$, for the Lagrangian trajectories associated with the solutions of the three-dimensional NSE. More precisely, suppose $u(x,t)$ is a Leray–Hopf weak solution of the 3D NSE. A Lagrangian trajectory is a solution $x(t)$ of the ODE $x' = u(x,t)$. I then proved in [32] that

$$\lim_{t\to\infty} x(t) = x_* \in \mathbb{R}^3 \text{ and } x(t) \sim x_* + \sum_{n=1}^{\infty} \zeta_n(t) e^{-\mu_n t} \text{ in } \mathbb{R}^3 \text{ as } t \to \infty,$$

where each $\zeta_n(t)$ is an $\mathbb{R}^3$-valued polynomial. Therefore, these asymptotic expansions provide very fine details for the long-time behaviors of the Lagrangian trajectories. This result may open up more systematic study of long-time Lagrangian dynamics, at least for decaying turbulence, which was out of reach previously.

2. Generalized Forchheimer flows of compressible fluids in porous media.

In the studies of porous media, Darcy's law is widely used to describe the fluid flows. This law is a linear equation between the pressure gradient and the fluid velocity. However, it is known that Darcy's law is not accurate in modeling fluid flows in many situations [2, 29, 53]. In fact, even in their original works [16,17], Darcy and Dupuit already noted the deviation from the linear relation. Forchheimer [29, 30], see also [2, 54, 58], later proposed three nonlinear equations to capture such deviations. They are called two-term, three-term and power laws. Compared to Darcy's law, Forchheimer equations are much less studied, particularly from the mathematical point of view. Also, mathematical papers on Forchheimer or related Brinkman-Forchheimer equations [7, 15, 31, 55–57, 59] are mostly for incompressible fluids.

My research involves the rigorous analysis of the so-called generalized Forchheimer equations. We propose those equations in [1, 39] to cover many possible variations of the Forchheimer equations. Their genuine nonlinearity makes them much more difficult to analyze than the well-known linear Darcy equation. In joint works [1, 8, 9, 11–13, 39–47], my collaborators and I extend significantly studies of the generalized Forchheimer equations to compressible fluids in porous media. We model and analyze single-phase compressible fluids, two-phase mixed fluids, fluids of mixed regimes (pre-Darcy, Darcy and post-Darcy), fluids in rotating porous media and anisotropic fluid flows. We derive various estimates for the solutions and its gradient by utilizing and improving many different techniques. We develop techniques for dynamical systems of degenerate parabolic equations, particularly asymptotic stability, long-time continuous dependence and structural stability. Our research also brings forth new nonlinear models that are still practical and physically relevant, and may stimulate the development of new PDE techniques.

Consider a porous medium with constant porosity $\phi \in (0,1)$ and constant permeability $k > 0$. We study a fluid flow in porous media with velocity $v(x,t) \in \mathbb{R}^n$, pressure $p(x,t) \in \mathbb{R}$ and density $\rho(x,t) \in \mathbb{R}_+ = [0,\infty)$, where $x \in \mathbb{R}^n$ $(n \geq 2)$ and $t \in \mathbb{R}$ are the spatial and time variables.

Mathematical papers on fluid flows in porous media usually deal with the equation $u_t = \Delta(u^m)$. It is based on Darcy's law $v = -\frac{k}{\mu}\nabla p$, where μ is the absolute viscosity. The generalized Forchheimer equation that we propose [1, 39–41] to replace Darcy's law is

$$g(|v|)v = -\nabla p, \text{ where } g(s) = a_0 + a_1 s^{\bar\alpha_1} + \cdots + a_N s^{\bar\alpha_N}, \quad \text{for } s \geq 0, \tag{2.1}$$

with the integer $N \geq 1$, the powers $\bar\alpha_0 = 0 < \bar\alpha_1 < \bar\alpha_2 < \ldots < \bar\alpha_N$ being real numbers, and coefficients $a_0, a_1, \ldots, a_N$ being positive constants. This equation covers all common cases of Forchheimer's two-term, three-term and power laws in literature. For (isothermal) slightly compressible fluids, we obtain

$$\phi \frac{\partial p}{\partial t} = \frac{1}{\varpi} \nabla \cdot (K(|\nabla p|)\nabla p), \tag{2.2}$$

where $\varpi > 0$ is a small constant compressibility. This equation is studied in a series papers for homogeneous porous media [1, 39–43, 47], for heterogeneous porous media when $a_i = a_i(x)$ [8, 9], and for different mixed regimes [10].

Using dimension analysis (Muskat [53]), we can also refine (2.1) as

$$g(\rho|v|)v = -\nabla p + \rho\vec{\mathbf{g}}, \tag{2.3}$$

where $\vec{\mathbf{g}}$ is the constant gravitational field. Then we derive a PDE

$$(u^\lambda)_t = \nabla \cdot (K(|\nabla u - cu^\ell \vec{\mathbf{g}}|)(\nabla u - cu^\ell \vec{\mathbf{g}})) \tag{2.4}$$

with constants $\lambda \in (0, 1]$, $\ell = 2\lambda$, $c > 0$. Compared with (2.2), equation (2.4) is double nonlinear and is much more complicated. In fact, even a much more general problem is studied in [11, 12]. It is analyzed rigorously in our works [11, 12], and requires the use and improvements of many techniques for degenerate/singular parabolic equations. Its study prompts us to investigate even more complex problems in [13, 14]. Namely, we study the dynamics of fluid flows in a porous medium rotated with a constant angular velocity $\Omega\vec{k}$, where $\Omega \geq 0$ is the constant angular speed, and $\vec{k}$ is a constant unit vector. The generalized Forchheimer equation in this rotating porous media is

$$g(\rho|v|)v + \frac{2\rho\Omega}{\phi}\vec{k} \times v + \rho\Omega^2\vec{k} \times (\vec{k} \times x) = -\nabla p + \rho\vec{\mathbf{g}}, \tag{2.5}$$

where x is the position in the rotating frame, $\Omega^2\vec{k}\times(\vec{k}\times x)$ is centripetal acceleration, and $(2\rho\Omega/\phi)\vec{k}\times v$ represents the Coriolis effects. In the case of slightly compressible fluids, we can derive a PDE for $u(x, t) \sim \rho(x, t)$:

$$\frac{\partial u}{\partial t} = \nabla \cdot \left(X\left(u, \nabla u + u^2\mathcal{Z}(x, t)\right)\right), \tag{2.6}$$

where $X = X(z, y) : \mathbb{R}_+ \times \mathbb{R}^3 \to \mathbb{R}^3$, $\mathcal{Z}(x, t) = -\mathcal{G}e_0(t) + \Omega^2\mathbf{J}^2 x$ with $\mathcal{G} = const.$, $|e_0(t)| = 1$, and the 3×3 matrix $\mathbf{J}$ represents the rotation. The new feature is the dependence of X on z. We find many important z-dependent properties of $X(z, y)$ in [14]. Based on these, many rigorous estimates are obtained in [11–14] by Celik, Kieu and myself.

Recently, Kieu and I study anisotropic Forchheimer equations in [44]. These equations have not been studied mathematically before. We discover essential structures of these nonlinear equations – under some specific conditions on the involved parameters. These conditions are, in fact, met by experimental data. We obtain the existence and uniqueness of steady states and establish their continuous dependence on the boundary data.

3. Asymptotic analysis for dynamical systems

3.1. **Attractors of abstract dynamical systems.** The dynamical systems arising in scientific and engineering problems usually depend on many parameters. While their long-time dynamics are captured by the global attractors, the dependence of these attractors on the parameters are hard to obtain. In [50], Olson, Robinson and myself study a family of general dynamical systems and their global attractors with parameters belonging to a metric space. We prove that these attractors are continuous, with respect to the Hausdorff distance, at values of the parameters that belong to a residual (large) set. The results are then extended to the non-autonomous systems, and then applied to the celebrated Lorenz and the two-dimensional Navier–Stokes systems [51]. Although the dynamics are sensitive to the change of the parameters, our results show certain degree of robustness of the attractors. With this knowledge, we hope that more reliable numerical schemes can be designed to capture more precisely the long-time dynamics of fluid flows.

3.2. **Asymptotic expansions for nonlinear systems of ordinary differential equations.** We develop the asymptotic expansion theory for general nonlinear ODEs.

3.2.1. *Systems with coherently decaying forcing functions.* Consider the following system of nonlinear ODEs in $\mathbb{C}^d$ or $\mathbb{R}^d$:

$$y' = -Ay + G(y) + f(t), \tag{3.1}$$

where A is a $d \times d$ and F is smooth in a neighborhood of the origin. In [5, 33], we establish the results of the form of Theorem 1.2 for the general equation (3.1).

3.2.2. *Systems with non-smooth nonlinearity.* We study the following ODE systems in $\mathbb{R}^d$

$$y' + Ay = F(y), \quad t > 0, \tag{3.2}$$

The goal is to obtain the asymptotic expansions for the solutions $y(t)$ of (3.2) even when F does not have a Taylor expansion about the origin. This lack of smoothness may cripple previous proofs to establish an asymptotic expansion for $y(t)$. However, we overcome this in our paper [6].

Theorem 3.1 (Cao–Hoang–Kieu [6])**.** *Let $y(t)$ be a non-trivial, decaying solution of (3.2). There exist polynomials q_n: $\mathbb{R} \to \mathbb{R}^d$ such that $y(t)$ has an asymptotic expansion*

$$y(t) \sim \sum_{n=1}^{\infty} q_n(t) e^{-\mu_n t} \text{ in } \mathbb{R}^d. \tag{3.3}$$

The techniques for the proof of Theorem 3.1 are then applied to obtain a new form of asymptotic expansion in our my work [35]. In that work we study equation (3.2) with a forcing function $f(t)$. The asymptotic expansion for a solution $y(t)$ seems to be out of reach. However, by setting a new *subordinate variable*, we are able to obtain a closed-form asymptotic expansion. More importantly, this new idea of subordinate variables may be applied to many other problems including the NSE.

3.2.3. *Asymptotic approximation for solutions of "genuine" nonlinear equations.* In the recent paper [37], I study equations of the form

$$y' = -H(y)Ay + G(y, t), \tag{3.4}$$

where $H(y)$ is a positively homogeneous of a positive degree, and $G(y, t)$ is a higher order term. This equation is more nonlinear than (3.1) and (3.2). It is proved in [37] that the decaying solution $y(t)$ behaves, as $t \to \infty$, like $t^{1/p}\xi$, for some $p > 0$, and an eigenvector ξ of the matrix A. In [34, 36] similar equations are studies and the asymptotic behavior of solutions near the finite extinction time or finite time of blow-up are established. These results are the first steps of a possible asymptotic expansion theory.

4. Future research

Each of the topics (i), (ii) and (iii)can be developed much more in depth and breadth. I list here only some ideas.

A. Related to the Navier–Stokes equations and viscous, incompressible fluids.

- Continue expanding the theory of asymptotic expansions and investigate their convergence.
- Study the statistical solutions in the case of decaying forces and apply to the turbulence theory.
- Study asymptotic expansions when the forces have noise. Develop the stochastic theory for the asymptotic expansions.
- Develop the new asymptotic theory in [18] for the full Navier–Stokes equations and other PDEs.
- Study the asymptotic expansions for the Lagrangian trajectories. Is there an associated normalization map in some sense, or a normal form theory? Establish the expansions for the case of non-potential forces.

B. Related to the Forchheimer flows in porous media.

- *Forchheimer flows in heterogeneous porous media.* This subject is barely touched in mathematical research due to the complex and sophisticated mathematics involved. We may need to utilize, modify or create new tools in analysis, theory of singular/degenerate partial differential equations.
- *Forchheimer flows with other effects.* Some examples are Klinkenberg's effect for gaseous flows and the dependence of the permeability on the pressure. The last one is known, for instance, for sea ice with the Darcy model, but the mathematical analysis is currently not available for the Forchheimer flows.
- *Multi-phase Forchheimer flows.* This is a challenging subject which my collaborators and I only managed to publish two papers [45, 46]. They deal with special steady states and the linearized problems. I plan to focus on the analysis of the non-stationary solutions.
- *Applications in agriculture.* I am interested in fluid dynamics in rhizosphere. My assessment is that our techniques are mature enough to analyze some nonlinear models without sacrificing the mathematical rigor.

C. Related to dynamical systems.

- For a family of parametric dynamical system, find the criteria for the existence of limiting dynamics. This may provide another view to the vanishing viscosity problem.
- Find applications, particularly in mathematical biology, for our results in [5, 6, 33–37].
- Develop the asymptotic analysis of solutions of many more types of nonlinear systems and time-dependent forces. What are their counterparts for PDE?

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